Occupation Times of BROWNian Bridges

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Abstract. In this note explicit formulæ for the density and the distribution of Occupation Times of a BROWNian Bridge $Y^{(\mu)}=\left\{Y_t^{(\mu)}\right\}_{0\le t\le 1}$, defined by $Y_t^{(\mu)}=W_t-t\cdot W_1-t\cdot\mu$ for all $0\le t\le 1$ and a standard BROWNian motion $W=\left\{W_t\right\}_{t\ge 0}$, are provided.

Key Words. BROWNian motion, WIENER process, BROWNian Bridge, Occupation Times.

Introduction. Let be $W=\left\{W_t\right\}_{t\ge 0}$ a standard BROWNian motion or WIENER process and $W^{(\mu)}=\left\{W_t^{(\mu)}\right\}_{t\ge 0}$ with $W_t^{(\mu)}=W_t-t\cdot\mu$ a WIENER process with linear drift, respectively. Then the induced process $Y^{(0)}=\left\{Y_t^{(0)}\right\}_{t=0}^{1}$ defined by $Y_t^{(0)}:=W_t-t\cdot W_1$ for all $0\le t\le 1$ is called a *BROWNian Bridge* and more generally the process $Y^{(\mu)}=\left\{Y_t^{(\mu)}\right\}_{t=0}^{1}$ defined by $Y_t^{(\mu)}:=W_t-t\cdot W_1-t\cdot\mu$ for all $0\le t\le 1$ is called a *BROWNian Bridge with linear drift*. Traditionally, a BROWNian Bridge $Y^{(0)}$ is introduced as a time-continuous process whose distribution is for all BOREL sets $\mathrm{C}\in\mathscr{C}\left[0;1\right]$ on the space $\mathbf{C}\left[0;1\right]$ the conditional probability $\mathbf{P}\left[Y^{(0)}\in\mathrm{C}\right]=\mathbf{P}\left[W\in\mathrm{C}\,\middle|\,W_1=0\right]$ of a WIENER process W given the condition $W_1=0$. It can be easily shown that this definition of a BROWNian Bridge by conditional probabilities is equivalent to the definition given above.

Furthermore *Occupation Times* $\Gamma_+\left(T;0\right)\left(z\right)$ of a certain continuous function $z:\left[0;T\right]\to\mathbb{R}$ are commonly defined by

$$\Gamma_+\left(T;0\right)\left(z\right):=\int_{t=0}^{T}\mathbf{1}\left[z\left(t\right)>0\right]\cdot dt,$$

and even more generally with respect to a certain level $\kappa\in\mathbb{R}$

$$\Gamma_+\left(T;\kappa\right)\left(z\right):=\int_{t=0}^{T}\mathbf{1}\left[z\left(t\right)>\kappa\right]\cdot dt.$$

For $\kappa\ne 0$ we denote by $\mathrm{T}_\kappa\left(z\right):=\inf\left\{\tau>0\,\middle|\,z\left(\tau\right)=\kappa\right\}$ the *First Passage Time* (FPT) of the level κ by z. After all, we think that the reader is familiar with the notation

$$\mathcal{N}(x)=\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{w=-\infty}^{x}\exp\left\{-\frac{w^2}{2}\right\}\cdot dw\,,\ x\in\mathbb{R}\,.$$

Distribution of Occupation Times of a BROWNian Bridge with Linear Drift. In this section we shall consider the general case of a BROWNian Bridge $Y^{(\mu)}=\left\{Y_t^{(\mu)}\right\}_{t=0}^{1}$ with linear drift, i.e.

$$Y_t^{(\mu)}=Y_t^{(0)}-t\cdot\mu=W_t-t\cdot W_1-t\cdot\mu\,,\ \mu\in\mathbb{R}\,,\ 0\le t\le 1\,,$$

and its occupation times $\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)=\int_{t=0}^{1}\mathbf{1}\left[Y_t^{(\mu)}>\kappa\right]\cdot dt$ in the time-interval $[0;1]$. Obviously, the distribution $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]$, $u\in\mathbb{R}$, is concentrated on the real interval $[0;1]$, i.e., $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]=0$ for all $u<0$, and $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]=1$ for all $u>1$. In the following we shall derive for $u\in[0;1]$, $dh=\lim_{h\to 0}[-h;h]$ and $dx=x+dh$ for $x\in\mathbb{R}$ the density and distribution of $\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)$.

For $0\le u\le 1$ we start the computation of the density $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\in du\right]$ and the distribution $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]$ with the case $\kappa>0$. Since $Y_0^{(\mu)}\equiv 0$ we have to pay attention to the fact that $\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)=0\right]$ may be larger than 0. Thus we obtain the following results.

$$\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)=0\right]=\mathbf{P}\left[\max_{0\le t\le 1}Y_t^{(\mu)}\le\kappa\right]=1-\mathbf{P}\left[\max_{0\le t\le 1}Y_t^{(\mu)}{}_t>\kappa\right],$$

where by GIRSANOV's theorem

$$\begin{aligned}
&\mathbf{P}\left[\max_{0\le t\le 1}Y_t^{(\mu)}>\kappa\right]=\mathbf{P}\left[\max_{0\le t\le 1}\left(W_t-t\cdot W_1-t\cdot\mu\right)>\kappa\right]=\\
&=\int_{x\in\mathbb{R}}\mathbf{P}\left[\max_{0\le t\le 1}\left(W_t-t\cdot x-t\cdot\mu\right)>\kappa;W_1\in x+dh\right]=\\
&=\int_{x\in\mathbb{R}}\mathbf{P}\left[\max_{0\le t\le 1}\left(W_t-t\cdot(x+\mu)\right)>\kappa;W_1-(x+\mu)\in-\mu+dh\right]=\\
&=\int_{x\in\mathbb{R}}\exp\left\{\frac{\mu^2-x^2}{2}\right\}\cdot\mathbf{P}\left[\max_{0\le t\le 1}W_t>\kappa;W_1\in-\mu+dh\right].
\end{aligned}$$

Now, if $\kappa<-\mu$, this yields $\mathbf{P}\left[\max_{0\le t\le 1}W_t>\kappa;W_1\in-\mu+dh\right]=\mathbf{P}\left[W_1\in-\mu+dh\right]$ and thus

$$\mathbf{P}\left[\max_{0\le t\le 1}Y_t^{(\mu)}>\kappa\right]=\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{x\in\mathbb{R}}\exp\left\{\frac{\mu^2-x^2}{2}\right\}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot dh=1\,,$$

such that $\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)=0\right]=0$.

Furthermore, if $\kappa \geq -\mu$, this yields $\mathbf{P}\left[\max_{0\leq t\leq 1} W_t > \kappa; W_1 \in -\mu + dh\right] = \mathbf{P}\left[W_1 \in 2\cdot\kappa+\mu+dh\right]$ by reflection principle and thus

$$\mathbf{P}\left[\max_{0\leq t\leq 1} Y_t^{(\mu)} > \kappa\right] = \frac{1}{\sqrt{2\cdot\pi}}\cdot \int_{x\in\mathbb{R}} \exp\left\{\frac{\mu^2-x^2}{2}\right\}\cdot\exp\left\{-\frac{\left(2\cdot\kappa+\mu\right)^2}{2}\right\}\cdot dh = \exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\},$$

such that $\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)=0\right]=\left(1-\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\right)$. Concluding we arrive at the result

(E-1) $$\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)=0\right]=\left(1-\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\right)\cdot\mathbf{1}\left[\kappa+\mu\geq 0\right].$$

Using the familiar techniques, especially GIRSANOV's theorem, we obtain for $0 < u < 1$

$$\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)\in du\right]=\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[Y_t^{(\mu)}>\kappa\right]\cdot dt\in du\right]=\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t-t\cdot W_1-t\cdot\mu>\kappa\right]\cdot dt\in du\right]=$$
$$=\int_{x\in\mathbb{R}}\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t-t\cdot x-t\cdot\mu>\kappa\right]\cdot dt\in du;W_1\in x+dh\right]=$$
$$=\int_{x\in\mathbb{R}}\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t-t\cdot\left(x+\mu\right)>\kappa\right]\cdot dt\in du;W_1-\left(x+\mu\right)\in-\mu+dh\right]=$$
$$=\int_{x\in\mathbb{R}}\mathbf{E}\left(\exp\left\{-\left(x+\mu\right)\cdot W_1-\frac{\left(x+\mu\right)^2}{2}\right\}\cdot\mathbf{1}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right]\right)=$$
$$=\int_{x\in\mathbb{R}}\exp\left\{\frac{\mu^2-x^2}{2}\right\}\cdot\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right].$$

Thus we only have to compute $\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right]$.

$$\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right]=\int_{\tau=0}^{1-u}\mathbf{P}\left[\mathrm{T}_{\kappa}\left(W\right)\in d\tau;\int_{t=\tau}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right].$$

Since $W_\tau=\kappa$ and with respect to the independence of increments we further obtain by applying the joint density formulæ for a WIENER process and its occupation times, $\mathbf{P}\left[\int_{t=0}^{\theta}\mathbf{1}\left[W_t>0\right]\cdot dt\in du;W_\theta\in\zeta+dh\right]$, and for its FPT the following integral representations,

$$\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[W_t>\kappa\right]\cdot dt\in du;W_1\in-\mu+dh\right]=$$

$$= \int_{\tau=0}^{1-u} \mathbf{P}\left[\mathrm{T}_\kappa(W) \in d\tau; \int_{t=\tau}^{1} \mathbf{1}[W_t - W_\tau > 0] \cdot dt \in du; W_1 - W_\tau \in -(\kappa+\mu) + dh \right] =$$

$$= \int_{\tau=0}^{1-u} \mathbf{P}\left[\int_{t=0}^{1-\tau} \mathbf{1}[W_t > 0] \cdot dt \in du; W_{1-\tau} \in -(\kappa+\mu) + dh \right] \cdot \mathbf{P}\left[\mathrm{T}_\kappa(W) \in d\tau \right] =$$

$$= \begin{cases} -\dfrac{\kappa+\mu}{2\cdot\pi} \cdot \displaystyle\int_{\tau=0}^{1-u} \int_{t=1-\tau-u}^{1-\tau} \frac{\exp\left\{ -\dfrac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)} \right\}}{\left[t\cdot(1-\tau-t) \right]^{\frac{3}{2}}} \cdot dt \cdot \frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \cdot du \cdot dh; & \kappa+\mu<0 \\ \dfrac{\kappa+\mu}{2\cdot\pi} \cdot \displaystyle\int_{\tau=0}^{1-u} \int_{t=u}^{1-\tau} \frac{\exp\left\{ -\dfrac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)} \right\}}{\left[t\cdot(1-\tau-t) \right]^{\frac{3}{2}}} \cdot dt \cdot \frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \cdot du \cdot dh; & \kappa+\mu>0 \end{cases}.$$

This yields to the following double integral representation.

$$\exp\left\{ -\frac{\mu^2}{2} \right\} \cdot \mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right) \in du \right] =$$

$$= \begin{cases} -\dfrac{\kappa+\mu}{\sqrt{2\cdot\pi}} \cdot \displaystyle\int_{\tau=0}^{1-u} \int_{t=1-\tau-u}^{1-\tau} \frac{\exp\left\{ -\dfrac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)} \right\}}{\left[t\cdot(1-\tau-t) \right]^{\frac{3}{2}}} \cdot dt \cdot \frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \cdot du; & \kappa+\mu<0 \\ \dfrac{\kappa+\mu}{\sqrt{2\cdot\pi}} \cdot \displaystyle\int_{\tau=0}^{1-u} \int_{t=u}^{1-\tau} \frac{\exp\left\{ -\dfrac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)} \right\}}{\left[t\cdot(1-\tau-t) \right]^{\frac{3}{2}}} \cdot dt \cdot \frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \cdot du; & \kappa+\mu>0 \end{cases}.$$

We shall now consider first the subcase $\kappa+\mu>0$. Then we obtain for the above double integral

$$\int_{\tau=0}^{1-u} \int_{t=u}^{1-\tau} \frac{1}{t^{\frac{3}{2}}} \cdot \left[\frac{\kappa+\mu}{\sqrt{2\cdot\pi}\cdot(1-\tau-t)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)} \right\} \cdot dt \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right],$$

and furthermore by substitution $y=1-t$ and $\upsilon=1-u$

$$\int_{\tau=0}^{\upsilon} \int_{y=\tau}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{\kappa+\mu}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right].$$

By commutation of the integration variables τ and y we arrive at the representation

$$\int_{y=0}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \left[\frac{\kappa+\mu}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)}\right\}\cdot dy \right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right].$$

Evidently, for this integral we just obtain by interpreting the appearing terms as densities of first passage times of the independent increments of a WIENER process

$$\int_{y=0}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \mathbf{P}\left[\mathrm{T}_{\kappa+\mu}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=0}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot\kappa+\mu}(W)\in dy\right]=$$

$$=\frac{2\cdot\kappa+\mu}{\sqrt{2\cdot\pi}}\cdot\int_{y=0}^{\upsilon}\frac{dy}{\left[(1-y)\cdot y\right]^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2\cdot y}\right\}.$$

Then by application of PECHTL (1999) [eq. (2), Remark 1.2] and remembering $\upsilon=1-u$ we arrive at the explicit representation for the density

(E-2) $$\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\in du\right]=$$

$$=2\cdot\left(1-(2\cdot\kappa+\mu)^2\right)\cdot\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\cdot\mathcal{N}\left(-(2\cdot\kappa+\mu)\cdot\sqrt{\frac{u}{1-u}}\right)\cdot du+$$

$$+2\cdot\frac{(2\cdot\kappa+\mu)}{\sqrt{2\cdot\pi}}\cdot\sqrt{\frac{1-u}{u}}\cdot\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\cdot\exp\left\{-\frac{u\cdot(2\cdot\kappa+\mu)^2}{2\cdot(1-u)}\right\}\cdot du\,.$$

By one straightforward integration step we finally obtain for all $0\le u\le 1$ the explicit representation of the distribution.

(E-3) $$\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]=\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)=0\right]+\int_{\omega=0}^{u}\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\in d\omega\right]=$$

$$=\left(1-\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\right)+\exp\left\{\frac{\mu^2}{2}\right\}\cdot\frac{2\cdot\kappa+\mu}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\int_{y=0}^{1-\omega}\frac{dy}{\left[(1-y)\cdot y\right]^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2\cdot y}\right\}\cdot d\omega=$$

$$=1+2\cdot\left(u\cdot\left(1-(2\cdot\kappa+\mu)^2\right)-1\right)\cdot\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\cdot\mathcal{N}\left(-(2\cdot\kappa+\mu)\cdot\sqrt{\frac{u}{1-u}}\right)+$$

$$+2\cdot\frac{2\cdot\kappa+\mu}{\sqrt{2\cdot\pi}}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\cdot\exp\left\{-\frac{u\cdot(2\cdot\kappa+\mu)^2}{2\cdot(1-u)}\right\}.$$

Moreover, we now consider the second subcase $\kappa+\mu<0$. Then we obtain for the above double integral

$$\int_{\tau=0}^{1\text{-}u}\int_{t=1-\tau-u}^{1-\tau}\frac{1}{t^{\frac{3}{2}}}\cdot\left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(1-\tau-t)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(1-\tau-t)}\right\}\cdot dt\right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right],$$

and furthermore by substitution $y=1-t$ and $\upsilon=1-u$

$$\int_{\tau=0}^{\upsilon}\int_{y=\tau}^{\tau+1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)}\right\}\cdot dy\right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right].$$

Now, we have to pay attention to the two-dimensional integration areas. For $0\le\upsilon\le\frac{1}{2}$ we have $\upsilon\le\tau+1-\upsilon$ such that the above integral can be represented by the sum of integrals

$$\int_{\tau=0}^{\upsilon}\int_{y=\tau}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)}\right\}\cdot dy\right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right]+$$
$$+\int_{\tau=0}^{\upsilon}\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)}\right\}\cdot dy\right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right]+$$
$$+\int_{\tau=0}^{\upsilon}\int_{y=1-\upsilon}^{\tau+1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)}\right\}\cdot dy\right]\cdot\left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau\right].$$

Interpreting the appearing terms as densities of first passage times of the independent increments of a WIENER process yields

$$\int_{\tau=0}^{\upsilon}\int_{y=\tau}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$
$$+\int_{\tau=0}^{\upsilon}\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$
$$+\int_{\tau=0}^{\upsilon}\int_{y=1-\upsilon}^{\tau+1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right].$$

By commutation of the integration variables τ and y we arrive at the representation

$$\int_{y=0}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{y}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$
$$+\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$

$$+\int_{y=1-\upsilon}^{1}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=y-1+\upsilon}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=0}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{y}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$

$$+\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]+$$

$$+\lim_{\omega\to1}\int_{y=1-\upsilon}^{\omega}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=y-1+\upsilon}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right].$$

We consider those three integrals successively. For the first one we obtain

$$\int_{y=0}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{y}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=0}^{\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy\right].$$

The second integral can be transformed as follows.

$$\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=0}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\mathrm{T}_{\kappa}(W)\in d\tau\right]=\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\mathrm{T}_{\kappa}(W)\in[0;\upsilon]\right]=$$

$$=\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\max_{0\le\tau\le\upsilon}W_{\tau}\ge\kappa\right]=$$

$$=\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy\right]-\int_{y=\upsilon}^{1-\upsilon}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\max_{0\le\tau\le\upsilon}W_{\tau}\le\kappa\right].$$

Finally, we obtain for the third integral

$$\int_{y=1-\upsilon}^{\omega}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=y-1+\upsilon}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W)\in dy-\tau\right]\cdot\mathbf{P}\left[\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=1-\upsilon}^{\omega}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\int_{\tau=y-1+\upsilon}^{\upsilon}\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\mathrm{T}_{\kappa}(W)\in d\tau\right]=$$

$$=\int_{y=1-\upsilon}^{\omega}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy;\mathrm{T}_{\kappa}(W)\in\left[y-1+\upsilon;\upsilon\right]\right]=$$

$$= \int_{y=1-\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left(\mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le\upsilon} W_\tau \ge \kappa \right] - \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_\tau \ge \kappa \right] \right) =$$

$$= \int_{y=1-\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left(\mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_\tau \le \kappa \right] - \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le\upsilon} W_\tau \le \kappa \right] \right).$$

Concluding these results we arrive at the following sum of integrals,

$$\int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy \right] +$$

$$+ \lim_{\omega\to 1} \left(\int_{y=1-\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_\tau \le \kappa \right] - \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le\upsilon} W_\tau \le \kappa \right] \right).$$

In the second case $\frac{1}{2} < \upsilon \le 1$ – where $1-\upsilon < \upsilon$ – we represent the integral in question by

$$\iint_B \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] =$$

$$= \iint_{B_1} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] +$$

$$+ \iint_{B_2} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] +$$

$$+ \iint_{B_3} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right],$$

where $B = B_1 + B_2 + B_3$ with $B := \{(\tau; y) | 0 \le \tau \le \upsilon; \tau \le y \le \tau + 1 - \upsilon\}$ is the union of the three pairwise disjoint sets

$$B_1 := \{(\tau; y) | 0 \le y \le 1-\upsilon; 0 \le \tau \le y\},$$
$$B_2 := \{(\tau; y) | 1-\upsilon < y \le \upsilon; y-1+\upsilon \le \tau \le y\},$$
$$B_3 := \{(\tau; y) | \upsilon < y \le 1; y-1+\upsilon \le \tau \le \upsilon\}.$$

Thus, we arrive at

$$\iint_B \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] =$$

$$= \int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] +$$

$$+ \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{y} \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] +$$

$$+ \int_{y=\upsilon}^{1} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{\upsilon} \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right].$$

Interpreting the appearing terms as densities of first passage times of the independent increments of a WIENER process this yields

$$\iint_B \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \left[\frac{-(\kappa+\mu)}{\sqrt{2\cdot\pi}\cdot(y-\tau)^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{(\kappa+\mu)^2}{2\cdot(y-\tau)} \right\} \cdot dy \right] \cdot \left[\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}} \cdot \exp\left\{ -\frac{\kappa^2}{2\cdot\tau} \right\} \cdot d\tau \right] =$$

$$= \int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] +$$

$$+ \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] +$$

$$+ \int_{y=\upsilon}^{1} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{\upsilon} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] =$$

$$= \int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] +$$

$$+ \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] +$$

$$+ \lim_{\omega\to 1} \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{\upsilon} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right].$$

Again, we consider those three integrals successively. For the first one we obtain

$$\int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau \right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau \right] =$$

$$= \int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy \right].$$

The second integral can be transformed as follows.

$$\int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau\right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau\right] =$$

$$= \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau\right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau\right] -$$

$$- \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=0}^{y-1+\upsilon} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau\right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau\right] =$$

$$= \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy\right] - \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \mathrm{T}_{\kappa}(W) \in [0; y-1+\upsilon]\right] =$$

$$= \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy\right] - \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_{\tau} \ge \kappa\right] =$$

$$= \int_{y=1-\upsilon}^{\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_{\tau} \le \kappa\right].$$

Finally, we obtain for the third integral

$$\int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \int_{\tau=y-1+\upsilon}^{\upsilon} \mathbf{P}\left[\mathrm{T}_{-(\kappa+\mu)}(W) \in dy-\tau\right] \cdot \mathbf{P}\left[\mathrm{T}_{\kappa}(W) \in d\tau\right] =$$

$$= \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \mathrm{T}_{\kappa}(W) \in [y-1+\upsilon; \upsilon]\right] =$$

$$= \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \mathrm{T}_{\kappa}(W) \in [0; \upsilon]\right] -$$

$$- \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \mathrm{T}_{\kappa}(W) \in [0; y-1+\upsilon]\right] =$$

$$= \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_{\tau} \le \kappa\right] -$$

$$- \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le\upsilon} W_{\tau} \le \kappa\right].$$

Concluding these results we arrive at the following sum of integrals,

$$\int_{y=0}^{1-\upsilon} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy\right] +$$

$$+ \lim_{\omega\to 1}\left(\int_{y=1-\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_{\tau} \le \kappa\right] - \int_{y=\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le\upsilon} W_{\tau} \le \kappa\right]\right).$$

Evidently, in both subcases for υ we arrive at the same integrals. Thus we have to determine the remaining limit for $\omega \to 1$. The first of those integrals can be transformed as follows.

$$\int_{y=1-\upsilon}^{\omega} \frac{1}{(1-y)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dy; \max_{0\le\tau\le y-1+\upsilon} W_\tau \le \kappa\right]^{w=y-1+\upsilon} =$$

$$= \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dw+1-\upsilon; \max_{0\le\tau\le w} W_\tau \le \kappa\right] =$$

$$= \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \mathbf{P}\left[\mathrm{T}_{-\mu}(W) \in dw+1-\upsilon; \max_{0\le\tau\le w} W_\tau \le \kappa; W_w \in dx\right] =$$

$$= \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \mathbf{P}\left[\mathrm{T}_{-(x+\mu)}(W) \in dh+1-\upsilon\right] \cdot \mathbf{P}\left[\max_{0\le\tau\le w} W_\tau \le \kappa; W_w \in dx\right] =$$

$$= \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \mathbf{P}\left[\mathrm{T}_{-(x+\mu)}(W) \in dh+1-\upsilon\right] \cdot \left(\mathbf{P}\left[W_w \in dx\right] - \mathbf{P}\left[W_w \in dx - 2\cdot\kappa\right]\right) =$$

$$= \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \mathbf{P}\left[\mathrm{T}_{-(x+\mu)}(W) \in dh+1-\upsilon\right] \cdot \mathbf{P}\left[W_w \in dx\right] -$$

$$- \int_{w=0}^{\omega-1+\upsilon} \frac{1}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \mathbf{P}\left[\mathrm{T}_{-(x+\mu)}(W) \in dh+1-\upsilon\right] \cdot \mathbf{P}\left[W_w \in dx - 2\cdot\kappa\right].$$

Defining $f(\xi;\tau) := \frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{\tau}} \cdot \exp\left\{-\frac{\xi^2}{2\cdot\tau}\right\}$ we obtain by integration by parts

$$\int_{w=0}^{\omega-1+\upsilon} \frac{dw}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \frac{\partial}{\partial\xi} f(x+\mu;1-\upsilon) \cdot f(x;w) \cdot dx -$$

$$- \int_{w=0}^{\omega-1+\upsilon} \frac{dw}{(\upsilon-w)^{\frac{3}{2}}} \cdot \int_{x=-\infty}^{\kappa} \frac{\partial}{\partial\xi} f(x+\mu;1-\upsilon) \cdot f(x-2\cdot\kappa;w) \cdot dx =$$

$$= \int_{x=-\infty}^{\kappa} f(x+\mu;1-\upsilon) \cdot \int_{w=0}^{\omega-1+\upsilon} \frac{dw}{(\upsilon-w)^{\frac{3}{2}}} \cdot \left\{\frac{\partial}{\partial\xi} f(x-2\cdot\kappa;w) - \frac{\partial}{\partial\xi} f(x;w)\right\} \cdot dx.$$

Now, we shall first consider the inner integral.

$$\int_{w=0}^{\omega-1+\upsilon} \frac{dw}{(\upsilon-w)^{\frac{3}{2}}} \cdot \left\{\frac{\partial}{\partial\xi} f(x-2\cdot\kappa;w) - \frac{\partial}{\partial\xi} f(x;w)\right\} =$$

$$= \frac{x}{\sqrt{2\cdot\pi}} \cdot \int_{w=0}^{\omega-1+\upsilon} \frac{\exp\left\{-\frac{x^2}{2\cdot w}\right\}}{(w\cdot(\upsilon-w))^{\frac{3}{2}}} \cdot dw - \frac{x-2\cdot\kappa}{\sqrt{2\cdot\pi}} \cdot \int_{w=0}^{\omega-1+\upsilon} \frac{\exp\left\{-\frac{(x-2\cdot\kappa)^2}{2\cdot w}\right\}}{(w\cdot(\upsilon-w))^{\frac{3}{2}}} \cdot dw.$$

Determining the integral $\frac{x}{\sqrt{2\cdot\pi}}\cdot\int\limits_{w=0}^{\omega-1+\upsilon}\frac{\exp\left\{-\frac{x^2}{2\cdot w}\right\}}{\left(w\cdot(\upsilon-w)\right)^{\frac{3}{2}}}\cdot dw$ we standardize this integral to

$$\frac{x}{\sqrt{2\cdot\pi}}\cdot\int\limits_{w=0}^{\omega-1+\upsilon}\frac{\exp\left\{-\frac{x^2}{2\cdot w}\right\}}{\left(w\cdot(\upsilon-w)\right)^{\frac{3}{2}}}\cdot dw=\frac{\left(\frac{x}{\sqrt{\upsilon}}\right)}{\sqrt{2\cdot\pi}\cdot\upsilon^{\frac{3}{2}}}\cdot\int\limits_{r=0}^{\frac{\omega-1+\upsilon}{\upsilon}}\frac{\exp\left\{-\frac{x^2}{2\cdot\upsilon\cdot r}\right\}}{\left(r\cdot(1-r)\right)^{\frac{3}{2}}}\cdot dr\,.$$

Furthermore, we have to pay attention to the case $0\le x\le\kappa$ and to the case $x<0$. In the case $0\le x\le\kappa$ we obtain

$$\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\left(1-\frac{x^2}{\upsilon}\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot(\omega-1+\upsilon)}}\right)+\frac{2\cdot x}{\sqrt{2\cdot\pi}\cdot\upsilon^2}\cdot\sqrt{\frac{\omega-1+\upsilon}{1-\omega}}\cdot\exp\left\{-\frac{x^2}{2\cdot(\omega-1+\upsilon)}\right\}.$$

In the case $x<0$ we obtain

$$\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\left(\frac{x^2}{\upsilon}-1\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left(x\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot(\omega-1+\upsilon)}}\right)+\frac{2\cdot x}{\sqrt{2\cdot\pi}\cdot\upsilon^2}\cdot\sqrt{\frac{\omega-1+\upsilon}{1-\omega}}\cdot\exp\left\{-\frac{x^2}{2\cdot(\omega-1+\upsilon)}\right\}.$$

Similarly, since $x-2\cdot\kappa<0$ for all $x\le\kappa$ the integral $\frac{x-2\cdot\kappa}{\sqrt{2\cdot\pi}}\cdot\int\limits_{w=0}^{\omega-1+\upsilon}\frac{\exp\left\{-\frac{(x-2\cdot\kappa)^2}{2\cdot w}\right\}}{\left(w\cdot(\upsilon-w)\right)^{\frac{3}{2}}}\cdot dw$ is equal to

$$\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\left(\frac{(x-2\cdot\kappa)^2}{\upsilon}-1\right)\cdot\exp\left\{-\frac{(x-2\cdot\kappa)^2}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left((x-2\cdot\kappa)\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot(\omega-1+\upsilon)}}\right)+$$
$$+\frac{2\cdot(x-2\cdot\kappa)}{\sqrt{2\cdot\pi}\cdot\upsilon^2}\cdot\sqrt{\frac{\omega-1+\upsilon}{1-\omega}}\cdot\exp\left\{-\frac{(x-2\cdot\kappa)^2}{2\cdot(\omega-1+\upsilon)}\right\}.$$

Concluding we have to consider the sums of the following integrals.

$$\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\int\limits_{x=0}^{\kappa}f\left(x+\mu;1-\upsilon\right)\cdot\left(1-\frac{x^2}{\upsilon}\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot(\omega-1+\upsilon)}}\right)\cdot dx+$$
$$+\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\int\limits_{x=-\infty}^{0}f\left(x+\mu;1-\upsilon\right)\cdot\left(\frac{x^2}{\upsilon}-1\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left(x\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot(\omega-1+\upsilon)}}\right)\cdot dx-$$

$$-\frac{2}{\upsilon^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f\left(x+\mu;1-\upsilon\right)\cdot\left(\frac{\left(x-2\cdot\kappa\right)^{2}}{\upsilon}-1\right)\cdot\exp\left\{-\frac{\left(x-2\cdot\kappa\right)^{2}}{2\cdot\upsilon}\right\}\cdot\mathcal{N}\left(\left(x-2\cdot\kappa\right)\cdot\sqrt{\frac{1-\omega}{\upsilon\cdot\left(\omega-1+\upsilon\right)}}\right)\cdot dx.$$

The remaining terms can be represented by the following integral,

$$\frac{2}{\sqrt{1-\omega}}\cdot\left(\frac{\omega-1+\upsilon}{\upsilon}\right)^{2}\cdot\int_{x=-\infty}^{\kappa} f\left(x+\mu;1-\upsilon\right)\cdot\left\{\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;\omega-1+\upsilon\right)-\frac{\partial}{\partial\xi}f\left(x;\omega-1+\upsilon\right)\right\}\cdot dx.$$

Now we transform the second of the two integrals.

$$\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}\left(W\right)\in dy;\max_{0\le\tau\le\upsilon}W_{\tau}\le\kappa\right]=$$

$$=\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa}\mathbf{P}\left[\mathrm{T}_{-\mu}\left(W\right)\in dy;W_{\upsilon}\in dx;\max_{0\le\tau\le\upsilon}W_{\tau}\le\kappa\right]=$$

$$=\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa}\mathbf{P}\left[\mathrm{T}_{-(x+\mu)}\left(W\right)\in dy-\upsilon\right]\cdot\mathbf{P}\left[W_{\upsilon}\in dx;\max_{0\le\tau\le\upsilon}W_{\tau}\le\kappa\right]=$$

$$=\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa}\mathbf{P}\left[\mathrm{T}_{-(x+\mu)}\left(W\right)\in dy-\upsilon\right]\cdot\left(\mathbf{P}\left[W_{\upsilon}\in dx\right]-\mathbf{P}\left[W_{\upsilon}\in dx-2\cdot\kappa\right]\right)=$$

$$=\int_{x=-\infty}^{\kappa}\mathbf{P}\left[W_{\upsilon}\in dx\right]\cdot\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(x+\mu)}\left(W\right)\in dy-\upsilon\right]-$$

$$-\int_{x=-\infty}^{\kappa}\mathbf{P}\left[W_{\upsilon}\in dx-2\cdot\kappa\right]\cdot\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(x+\mu)}\left(W\right)\in dy-\upsilon\right].$$

Thus we first determine the inner integral

$$\int_{y=\upsilon}^{\omega}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-(x+\mu)}\left(W\right)\in dy-\upsilon\right]=-\frac{x+\mu}{\sqrt{2\cdot\pi}}\cdot\int_{y=\upsilon}^{\omega}\frac{\exp\left\{-\frac{\left(x+\mu\right)^{2}}{2\cdot\left(y-\upsilon\right)}\right\}}{\left[\left(1-y\right)\cdot\left(y-\upsilon\right)\right]^{\frac{3}{2}}}\cdot dy\overset{w=y-\upsilon}{=}$$

$$=-\frac{x+\mu}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{\omega-\upsilon}\frac{\exp\left\{-\frac{\left(x+\mu\right)^{2}}{2\cdot w}\right\}}{\left[\left(1-\upsilon-w\right)\cdot w\right]^{\frac{3}{2}}}\cdot dw\overset{v=\frac{w}{1-\upsilon}}{=}-\frac{1}{\left(1-\upsilon\right)^{\frac{3}{2}}}\frac{x+\mu}{\sqrt{1-\upsilon}}\cdot\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{v=0}^{\frac{\omega-\upsilon}{1-\upsilon}}\frac{\exp\left\{-\frac{\left(x+\mu\right)^{2}}{2\cdot\left(1-\upsilon\right)\cdot v}\right\}}{\left[\left(1-v\right)\cdot v\right]^{\frac{3}{2}}}\cdot dv.$$

Since $x+\mu<0$ for all $x\le\kappa$ we obtain by application of eq. (A-2)

$$\frac{2}{(1-\upsilon)^{\frac{3}{2}}}\cdot\left(1-\frac{(x+\mu)^2}{1-\upsilon}\right)\cdot\exp\left\{-\frac{(x+\mu)^2}{2\cdot(1-\upsilon)}\right\}\cdot\mathcal{N}\left((x+\mu)\cdot\sqrt{\frac{1-\omega}{(1-\upsilon)\cdot(\omega-\upsilon)}}\right)-$$
$$-\frac{2\cdot(x+\mu)}{\sqrt{2\cdot\pi}\cdot(1-\upsilon)^2}\cdot\sqrt{\frac{\omega-\upsilon}{1-\omega}}\cdot\exp\left\{-\frac{(x+\mu)^2}{2\cdot(\omega-\upsilon)}\right\}.$$

With this result we now consider the integrals

$$\frac{2}{(1-\upsilon)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f(x;\upsilon)\cdot\left(1-\frac{(x+\mu)^2}{1-\upsilon}\right)\cdot\exp\left\{-\frac{(x+\mu)^2}{2\cdot(1-\upsilon)}\right\}\cdot\mathcal{N}\left((x+\mu)\cdot\sqrt{\frac{1-\omega}{(1-\upsilon)\cdot(\omega-\upsilon)}}\right)\cdot dx-$$
$$-\frac{2}{(1-\upsilon)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f(x-2\cdot\kappa;\upsilon)\cdot\left(1-\frac{(x+\mu)^2}{1-\upsilon}\right)\cdot\exp\left\{-\frac{(x+\mu)^2}{2\cdot(1-\upsilon)}\right\}\cdot\mathcal{N}\left((x+\mu)\cdot\sqrt{\frac{1-\omega}{(1-\upsilon)\cdot(\omega-\upsilon)}}\right)\cdot dx$$

and

$$\frac{2}{\sqrt{1-\omega}}\cdot\left(\frac{\omega-\upsilon}{1-\upsilon}\right)^2\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f(x+\mu;\omega-\upsilon)\cdot\left\{f(x;\upsilon)-f(x-2\cdot\kappa;\upsilon)\right\}\cdot dx=$$
$$=\frac{2}{\sqrt{1-\omega}}\cdot\left(\frac{\omega-\upsilon}{1-\upsilon}\right)^2\cdot f(\kappa+\mu;\omega-\upsilon)\cdot\left\{f(\kappa;\upsilon)-f(-\kappa;\upsilon)\right\}-$$
$$-\frac{2}{\sqrt{1-\omega}}\cdot\left(\frac{\omega-\upsilon}{1-\upsilon}\right)^2\cdot\int_{x=-\infty}^{\kappa} f(x+\mu;\omega-\upsilon)\cdot\left\{\frac{\partial}{\partial\xi}f(x;\upsilon)-\frac{\partial}{\partial\xi}f(x-2\cdot\kappa;\upsilon)\right\}\cdot dx=$$
$$=\frac{2}{\sqrt{1-\omega}}\cdot\left(\frac{\omega-\upsilon}{1-\upsilon}\right)^2\cdot\int_{x=-\infty}^{\kappa} f(x+\mu;\omega-\upsilon)\cdot\left\{\frac{\partial}{\partial\xi}f(x-2\cdot\kappa;\upsilon)-\frac{\partial}{\partial\xi}f(x;\upsilon)\right\}\cdot dx.$$

Now, for $\omega\to 1$ the difference of the corresponding second integrals converges to 0 such that we only have to pay attention to the first ones. With respect to $\omega\to 1$ those integrals reduce to the following sum, using

$$\frac{\partial}{\partial\xi}\left[\frac{\xi}{\sqrt{\tau^3}}\cdot\exp\left\{-\frac{\xi^2}{2\cdot\tau}\right\}\right]=\frac{1}{\sqrt{\tau^3}}\cdot\left(1-\frac{\xi^2}{\tau}\right)\cdot\exp\left\{-\frac{\xi^2}{2\cdot\tau}\right\},$$

and applying integration by parts

$$\frac{1}{\upsilon^{\frac{3}{2}}}\cdot\int_{x=0}^{\kappa} f(x+\mu;1-\upsilon)\cdot\left(1-\frac{x^2}{\upsilon}\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot dx+$$
$$+\frac{1}{\upsilon^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{0} f(x+\mu;1-\upsilon)\cdot\left(\frac{x^2}{\upsilon}-1\right)\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}\cdot dx-$$

$$-\frac{1}{\upsilon^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f\left(x+\mu;1-\upsilon\right)\cdot\left(\frac{\left(x-2\cdot\kappa\right)^2}{\upsilon}-1\right)\cdot\exp\left\{-\frac{\left(x-2\cdot\kappa\right)^2}{2\cdot\upsilon}\right\}\cdot dx-$$

$$-\frac{1}{\left(1-\upsilon\right)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f\left(x;\upsilon\right)\cdot\left(1-\frac{\left(x+\mu\right)^2}{1-\upsilon}\right)\cdot\exp\left\{-\frac{\left(x+\mu\right)^2}{2\cdot\left(1-\upsilon\right)}\right\}\cdot dx+$$

$$+\frac{1}{\left(1-\upsilon\right)^{\frac{3}{2}}}\cdot\int_{x=-\infty}^{\kappa} f\left(x-2\cdot\kappa;\upsilon\right)\cdot\left(1-\frac{\left(x+\mu\right)^2}{1-\upsilon}\right)\cdot\exp\left\{-\frac{\left(x+\mu\right)^2}{2\cdot\left(1-\upsilon\right)}\right\}\cdot dx=$$

$$=f\left(\kappa+\mu;1-\upsilon\right)\cdot\frac{\kappa}{\sqrt{\upsilon^3}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\upsilon}\right\}+\sqrt{2\cdot\pi}\cdot\int_{x=0}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x;\upsilon\right)\cdot dx-$$

$$-\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{0}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x;\upsilon\right)\cdot dx-$$

$$-f\left(\kappa+\mu;1-\upsilon\right)\cdot\frac{\kappa}{\sqrt{\upsilon^3}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\upsilon}\right\}+\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;\upsilon\right)\cdot dx-$$

$$-f\left(\kappa;\upsilon\right)\cdot\frac{\left(\kappa+\mu\right)}{\sqrt{\left(1-\upsilon\right)^3}}\cdot\exp\left\{-\frac{\left(\kappa+\mu\right)^2}{2\cdot\upsilon}\right\}-\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x;\upsilon\right)\cdot dx+$$

$$+f\left(\kappa;\upsilon\right)\cdot\frac{\left(\kappa+\mu\right)}{\sqrt{\left(1-\upsilon\right)^3}}\cdot\exp\left\{-\frac{\left(\kappa+\mu\right)^2}{2\cdot\upsilon}\right\}+\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;\upsilon\right)\cdot dx=$$

$$=2\cdot\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;\upsilon\right)\cdot dx-$$

$$-2\cdot\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{0}\frac{\partial}{\partial\xi}f\left(x+\mu;1-\upsilon\right)\cdot\frac{\partial}{\partial\xi}f\left(x;\upsilon\right)\cdot dx\,.$$

By re-substitution $1-\upsilon=u$ we arrive at

$$\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathbf{P}\left[\Gamma_+\left(1;\kappa\right)\left(Y^{(\mu)}\right)\in du\right]=\int_{y=0}^{u}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}\left(W\right)\in dy\right]\cdot du+$$

$$+2\cdot\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;u\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-u\right)\cdot dx\cdot du-$$

$$-2\cdot\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{0}\frac{\partial}{\partial\xi}f\left(x+\mu;u\right)\cdot\frac{\partial}{\partial\xi}f\left(x;1-u\right)\cdot dx\cdot du\,,$$

and consequently, since we have to compute the distribution function

$$\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathbf{P}\left[\Gamma_+\left(1;\kappa\right)\left(Y^{(\mu)}\right)\le u\right]=\int_{w=0}^{u}\int_{y=0}^{w}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}\left(W\right)\in dy\right]\cdot dw+$$

$$+2\cdot\sqrt{2\cdot\pi}\cdot\int_{w=0}^{u}\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-w\right)\cdot dx\cdot dw-$$

$$-2\cdot\sqrt{2\cdot\pi}\cdot\int_{w=0}^{u}\int_{x=-\infty}^{0}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x;1-w\right)\cdot dx\cdot dw=$$

$$=\int_{w=0}^{u}\int_{y=0}^{w}\frac{1}{\left(1-y\right)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}\left(W\right)\in dy\right]\cdot dw+J\left(u,\kappa;\mu\right)-J\left(u,0;\mu\right),$$

where $J\left(u,\kappa;\mu\right)$ is defined and explicitly computed in Appendix A. Thus, applying equations (A-7) and (A-8) we arrive at the following closed form representation of $\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)\le u\right]$ in the case $\kappa+\mu<0$,

(E-4) $$\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)\le u\right]=1-\mathcal{N}\left(\frac{\mu\cdot u-\left(\kappa+\mu\right)}{\sqrt{u\cdot\left(1-u\right)}}\right)-$$

$$-\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+2\cdot\left(1-\left(2\cdot\kappa+\mu\right)^{2}\right)\cdot u\cdot\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+2\cdot\left(2\cdot\kappa+\mu\right)\cdot\left(\kappa+\mu\right)\cdot\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+\frac{2}{\sqrt{2\cdot\pi}}\cdot\left(2\cdot\kappa+\mu\right)\cdot\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\sqrt{u\cdot\left(1-u\right)}\cdot\exp\left\{-\frac{\left(\left(\kappa+\mu\right)-\left(2\cdot\kappa+\mu\right)\cdot u\right)^{2}}{2\cdot u\cdot\left(1-u\right)}\right\}.$$

REMARK. For the special case $\kappa+\mu=0\Leftrightarrow\kappa=-\mu$ the two formulæ (E-3) and (E-4) coincide. Respectively, a general formula for the distribution $\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)\le u\right]$ can be provided,

$$\mathbf{P}\left[\Gamma_{+}\left(1;\kappa\right)\left(Y^{(\mu)}\right)\le u\right]=$$

$$=1-\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\exp\left\{2\cdot\kappa\cdot\left(\kappa+\mu\right)^{-}\right\}\cdot\mathcal{N}\left(\frac{\left(2\cdot\left(\kappa+\mu\right)^{-}-\left(2\cdot\kappa+\mu\right)\right)\cdot u-\left(\kappa+\mu\right)^{-}}{\sqrt{u\cdot\left(1-u\right)}}\right)-$$

$$-\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)^{-}-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+2\cdot\left(1-\left(2\cdot\kappa+\mu\right)^{2}\right)\cdot u\cdot\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)^{-}-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+2\cdot\left(2\cdot\kappa+\mu\right)\cdot\left(\kappa+\mu\right)^{-}\cdot\exp\left\{-2\cdot\kappa\cdot\left(\kappa+\mu\right)\right\}\cdot\mathcal{N}\left(\frac{\left(\kappa+\mu\right)^{-}-\left(2\cdot\kappa+\mu\right)\cdot u}{\sqrt{u\cdot\left(1-u\right)}}\right)+$$

$$+\frac{2}{\sqrt{2\cdot\pi}}\cdot(2\cdot\kappa+\mu)\cdot\exp\{-2\cdot\kappa\cdot(\kappa+\mu)\}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\left\{-\frac{\left((\kappa+\mu)^{-}-(2\cdot\kappa+\mu)\cdot u\right)^2}{2\cdot u\cdot(1-u)}\right\}.$$

Using $-Y^{(\mu)}\overset{\mathscr{D}}{=}Y^{(-\mu)}$ for $\kappa<0$ the distribution of $\Gamma_+(1;\kappa)(Y)$ can be computed via

$$\mathbf{P}\left[\Gamma_+(1;\kappa)\left(Y^{(\mu)}\right)\le u\right]=\mathbf{P}\left[\Gamma_+(1;\kappa)\left(-Y^{(-\mu)}\right)\le u\right]=\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[Y_t^{(-\mu)}<-\kappa\right]\cdot dt\le u\right]=$$

$$=\mathbf{P}\left[\int_{t=0}^{1}\mathbf{1}\left[Y_t^{(-\mu)}\le-\kappa\right]\cdot dt\le u\right]=\mathbf{P}\left[\Gamma_+(1;-\kappa)\left(Y^{(-\mu)}\right)\ge 1-u\right]=1-\mathbf{P}\left[\Gamma_+(1;-\kappa)\left(Y^{(-\mu)}\right)<1-u\right].$$

Uniform Distribution of the Occupation Times of a Standard BROWNian Bridge. As a special case of formula (E-4) we obtain that the occupation times $\Gamma_+(1;0)\left(Y^{(0)}\right)$ of a standard BROWNian Bridge, i.e. with zero-drift, are *uniformly distributed*, i.e.

$$\text{(E-5)}\qquad \mathbf{P}\left[\Gamma_+(1;0)(W)\le u\right]=u$$

holds for all $0\le u\le 1$. This result may appear to be surprising, however, it is well known independently from the above proof. Moreover, it was even implicitly used for our proof: BILLINGSLEY (1968) derived the joint density formula $\mathbf{P}\left[\Gamma_+(1;0)(W)\in du;W_1\in dx\right]$ considering a random walk process. In this context the result (B-2)

$$\mathbf{P}\left[\Gamma_+(1;0)(W)\in u+dh\middle|W_1=0\right]=du\,,\ 0\le u\le 1,$$

was proved by elementary methods and by application of DONSKER's central limit theorem for a WIENER process [cf. Appendix B]. Thus, with respect to eq. (B-1) we obtain (E-5) immediately from (B-2).

APPENDIX A

Computation of a Certain Type of Integral. Strongly connected to the distribution of FPT of a BROWNian Motion is the following type of integral,

$$I(u;x,y):=\frac{x\cdot y}{2\cdot\pi}\cdot\int_{t=0}^{u}\frac{\exp\left\{-\frac{x^2}{2\cdot t}-\frac{y^2}{2\cdot(u-t)}\right\}}{\left[t\cdot(u-t)\right]^{\frac{3}{2}}}\cdot dt\,,\ x,y>0\,,\ u\ge 0\,.$$

As well known the density of the FPT $\mathrm{T}_\kappa(W)$ to a certain level $\kappa(>0)$ of a BROWNian Motion at time $\tau(\geq 0)$ is given by

$$\mathbf{P}\left[\mathrm{T}_\kappa(W)\in d\tau\right]=\frac{\kappa}{\sqrt{2\cdot\pi}\cdot\tau^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\kappa^2}{2\cdot\tau}\right\}\cdot d\tau .$$

Then, $I(u;x,y)$ can be represented by

$$I(u;x,y)=\int_{t=0}^{u}\mathbf{P}\left[\mathrm{T}_x(W)\in dt\right]\cdot\frac{\mathbf{P}\left[\mathrm{T}_y\left(W^{(t)}\right)\in du-t\right]}{du},$$

where $W=\{W_\tau\}_{\tau\geq 0}$ and $W^{(t)}=\left\{W_\tau^{(t)}\right\}_{\tau\geq 0}$ are two independent WIENER processes for each $0\leq t\leq u$. Since the increments of WIENER processes are independent, we can represent $W_\tau^{(t)}$ by $W_\tau^{(t)}=W_{\tau+t}-W_t$, which immediately – without any further computation – yields to

(A-1) $$I(u;x,y)=\frac{1}{du}\int_{t=0}^{u}\mathbf{P}\left[\mathrm{T}_x(W)\in dt;\mathrm{T}_{x+y}(W)\in du\right]=\frac{\mathbf{P}\left[\mathrm{T}_{x+y}(W)\in du\right]}{du}=$$
$$=\frac{x+y}{\sqrt{2\cdot\pi}\cdot u^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(x+y)^2}{2\cdot u}\right\}.$$

Moreover, to demonstrate the advantages of this interpretation we shall compute that rather complicated integral without the argument of independence of increments by common integration techniques.

$$I(u;x,y)=\frac{x\cdot y}{2\cdot\pi}\cdot\int_{t=0}^{u}\frac{\exp\left\{-\frac{x^2}{2\cdot t}-\frac{y^2}{2\cdot(u-t)}\right\}}{\left[t\cdot(u-t)\right]^{\frac{3}{2}}}\cdot dt\overset{t=u\cdot\tau,\xi=\frac{x}{\sqrt{u}},\upsilon=\frac{y}{\sqrt{u}}}{=}\frac{1}{u}\cdot\frac{\xi\cdot\upsilon}{2\cdot\pi}\cdot\int_{\tau=0}^{1}\frac{\exp\left\{-\frac{\xi^2}{2\cdot\tau}-\frac{\upsilon^2}{2\cdot(1-\tau)}\right\}}{\left[\tau\cdot(1-\tau)\right]^{\frac{3}{2}}}\cdot d\tau.$$

By further substitution $z=\frac{\xi}{\sqrt{\tau}}$ we obtain

$$\frac{1}{\pi\cdot u}\cdot\int_{z=\xi}^{\infty}\exp\left\{-\frac{z^2}{2}\right\}\cdot\frac{\upsilon\cdot z^3}{\left(z^2-\xi^2\right)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\upsilon^2\cdot z^2}{2\cdot\left(z^2-\xi^2\right)}\right\}\cdot dz=$$

$$=\exp\left\{-\frac{\xi^2+\upsilon^2}{2}\right\}\cdot\frac{1}{\pi\cdot u}\cdot\int_{z=\xi}^{\infty}\frac{\upsilon\cdot z^3}{\left(z^2-\xi^2\right)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{z^2-\xi^2}{2}\right\}\cdot\exp\left\{-\frac{\xi^2\cdot\upsilon^2}{2\cdot\left(z^2-\xi^2\right)}\right\}\cdot dz=$$

$$= \exp\left\{-\frac{\xi^2+\upsilon^2}{2}\right\}\cdot\frac{1}{\pi\cdot u}\cdot\int_{z=\xi}^{\infty}\frac{\upsilon\cdot z}{\left(z^2-\xi^2\right)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{z^2-\xi^2}{2}\right\}\cdot\exp\left\{-\frac{\xi^2\cdot\upsilon^2}{2\cdot\left(z^2-\xi^2\right)}\right\}\cdot dz+$$

$$+\exp\left\{-\frac{\xi^2+\upsilon^2}{2}\right\}\cdot\frac{\xi^2}{\pi\cdot u}\cdot\int_{z=\xi}^{\infty}\frac{\upsilon\cdot z}{\left(z^2-\xi^2\right)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{z^2-\xi^2}{2}\right\}\cdot\exp\left\{-\frac{\xi^2\cdot\upsilon^2}{2\cdot\left(z^2-\xi^2\right)}\right\}\cdot dz \overset{w=\sqrt{z^2-\xi^2},\,v=\frac{\xi\cdot\upsilon}{\sqrt{z^2-\xi^2}}}{=}$$

$$= \exp\left\{-\frac{\xi^2+\upsilon^2}{2}\right\}\cdot\frac{\upsilon}{\pi\cdot u}\cdot\int_{w=0}^{\infty}\exp\left\{-\frac{w^2}{2}\right\}\cdot\exp\left\{-\frac{\xi^2\cdot\upsilon^2}{2\cdot w^2}\right\}\cdot dw+$$

$$+\exp\left\{-\frac{\xi^2+\upsilon^2}{2}\right\}\cdot\frac{\xi}{\pi\cdot u}\cdot\int_{v=0}^{\infty}\exp\left\{-\frac{v^2}{2}\right\}\cdot\exp\left\{-\frac{\xi^2\cdot\upsilon^2}{2\cdot v^2}\right\}\cdot dv=$$

$$=\frac{\xi+\upsilon}{u}\cdot\exp\left\{-\frac{\left(\xi+\upsilon\right)^2}{2}\right\}\cdot\frac{1}{\pi}\cdot\int_{w=0}^{\infty}\exp\left\{-\frac{1}{2}\cdot\left(w-\frac{\xi\cdot\upsilon}{w}\right)^2\right\}\cdot dw.$$

Since the following relations

$$\frac{2}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{\infty}\exp\left\{-\frac{1}{2}\cdot\left(w-\frac{\xi\cdot\upsilon}{w}\right)^2\right\}\cdot dw=$$

$$=\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{\infty}\left(1+\frac{\xi\cdot\upsilon}{w^2}\right)\cdot\exp\left\{-\frac{1}{2}\cdot\left(w-\frac{\xi\cdot\upsilon}{w}\right)^2\right\}\cdot dw+\frac{e^{2\cdot\xi\cdot\upsilon}}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{\infty}\left(1-\frac{\xi\cdot\upsilon}{w^2}\right)\cdot\exp\left\{-\frac{1}{2}\cdot\left(w+\frac{\xi\cdot\upsilon}{w}\right)^2\right\}\cdot dw=$$

$$=\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{\zeta=-\infty}^{\infty}\exp\left\{-\frac{1}{2}\cdot\zeta^2\right\}\cdot d\zeta=1$$

hold, we finally arrive at

$$I\left(u;x,y\right)=\frac{\xi+\upsilon}{\sqrt{2\cdot\pi}\cdot u}\cdot\exp\left\{-\frac{\left(\xi+\upsilon\right)^2}{2}\right\}=\frac{x+y}{\sqrt{2\cdot\pi}\cdot u^{\frac{3}{2}}}\cdot\exp\left\{-\frac{\left(x+y\right)^2}{2\cdot u}\right\}.\square$$

Explicit Computation of the Distribution Integral. With respect to the explicit formulæ for standard BROWNIAN Motion and its Occupation Times in PECHTL (1999) [eq. (2), Remark 1.2] we immediately obtain for $x>0$ and $0\le\upsilon\le1$ the following representation.

$$\text{(A-2)}\quad \frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{y=0}^{\upsilon}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot\left(1-y\right)\right]^{\frac{3}{2}}}\cdot dy=$$

$$=2\cdot\left(1-x^2\right)\cdot\exp\left\{-\frac{x^2}{2}\right\}\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{1-\upsilon}{\upsilon}}\right)+\frac{2\cdot x}{\sqrt{2\cdot\pi}}\cdot\sqrt{\frac{\upsilon}{1-\upsilon}}\cdot\exp\left\{-\frac{x^2}{2\cdot\upsilon}\right\}.$$

With respect to explicit formulæ of the distribution integral we have to compute for arbitrary $0\le u\le1$ the double integral

$$\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\int_{y=0}^{1-\omega}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy\cdot d\omega\,.$$

By partial integration we reduce this double integral to two single integrals.

$$\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\int_{y=0}^{1-\omega}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy\cdot d\omega=u\cdot\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{y=0}^{1-u}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy+\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\frac{\exp\left\{-\frac{x^2}{2\cdot(1-\omega)}\right\}}{(1-\omega)^{\frac{3}{2}}\cdot\omega^{\frac{1}{2}}}\cdot d\omega.$$

By eq. (A-2) we obtain for the first of those two integrals

$$u\cdot\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{y=0}^{1-u}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy=$$
$$=2\cdot u\cdot\left(1-x^2\right)\cdot\exp\left\{-\frac{x^2}{2}\right\}\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{u}{1-u}}\right)+\frac{2\cdot x}{\sqrt{2\cdot\pi}}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\left\{-\frac{x^2}{2\cdot(1-u)}\right\}.$$

The second integral is directly computed by familiar transformations.

$$\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\frac{\exp\left\{-\frac{x^2}{2\cdot(1-\omega)}\right\}}{(1-\omega)^{\frac{3}{2}}\cdot\omega^{\frac{1}{2}}}\cdot d\omega\overset{\rho=1-\omega}{=}\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\rho=1-u}^{1}\frac{\exp\left\{-\frac{x^2}{2\cdot\rho}\right\}}{\rho^{\frac{3}{2}}\cdot(1-\rho)^{\frac{1}{2}}}\cdot d\rho\overset{\zeta=\frac{1}{\sqrt{\rho}}}{=}$$
$$=2\cdot\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\zeta=1}^{\frac{1}{\sqrt{1-u}}}\frac{\zeta\cdot\exp\left\{-\frac{x^2}{2}\cdot\zeta^2\right\}}{\left(\zeta^2-1\right)^{\frac{1}{2}}}\cdot d\zeta=2\cdot\exp\left\{-\frac{x^2}{2}\right\}\cdot\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\zeta=1}^{\frac{1}{\sqrt{1-u}}}\frac{\zeta\cdot\exp\left\{-\frac{x^2}{2}\cdot\left(\zeta^2-1\right)\right\}}{\left(\zeta^2-1\right)^{\frac{1}{2}}}\cdot d\zeta.$$

Now by substitution $\xi=\sqrt{\zeta^2-1}$ and $d\xi=\frac{\zeta\cdot d\zeta}{\sqrt{\zeta^2-1}}$ the last integral can be further transformed.

$$2\cdot\exp\left\{-\frac{x^2}{2}\right\}\cdot\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\xi=0}^{\sqrt{\frac{u}{1-u}}}\exp\left\{-\frac{x^2}{2}\cdot\xi^2\right\}\cdot d\xi\overset{\nu=x\cdot\xi}{=}\exp\left\{-\frac{x^2}{2}\right\}\cdot\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{\nu=-x\cdot\sqrt{\frac{u}{1-u}}}^{x\cdot\sqrt{\frac{u}{1-u}}}\exp\left\{-\frac{\nu^2}{2}\right\}\cdot d\nu=$$
$$=\exp\left\{-\frac{x^2}{2}\right\}\cdot\left(1-2\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{u}{1-u}}\right)\right).$$

Summarizing all those single results we obtain

(A-3) $$\frac{x}{\sqrt{2\cdot\pi}}\cdot\int_{\omega=0}^{u}\int_{y=0}^{1-\omega}\frac{\exp\left\{-\frac{x^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy\cdot d\omega=$$
$$=\exp\left\{-\frac{x^2}{2}\right\}+2\cdot\left(u\cdot\left(1-x^2\right)-1\right)\cdot\exp\left\{-\frac{x^2}{2}\right\}\cdot\mathcal{N}\left(-x\cdot\sqrt{\frac{u}{1-u}}\right)+\frac{2\cdot x}{\sqrt{2\cdot\pi}}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\left\{-\frac{x^2}{2\cdot(1-u)}\right\}.$$

Equation (A-3) is the most important auxiliary result for the distribution's computation in the case $\kappa+\mu>0$. For the considerably sophisticated case $\kappa+\mu<0$ explicit solutions of some more types of non-trivial integrals have to be provided. We start with the computation of

$$\int_{w=0}^{u}\int_{y=0}^{w}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy\right]\cdot dw=-\frac{\mu}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{u}\int_{y=0}^{w}\frac{\exp\left\{-\frac{\mu^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy\cdot dw,$$

where $\mu<0$. Again, we use integration by parts and apply then eq. (A-2),

$$-\frac{\mu}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{u}\int_{y=0}^{w}\frac{\exp\left\{-\frac{\mu^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy\cdot dw=-\frac{\mu}{\sqrt{2\cdot\pi}}\cdot u\cdot\int_{y=0}^{u}\frac{\exp\left\{-\frac{\mu^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy+\frac{\mu}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{u}\frac{\exp\left\{-\frac{\mu^2}{2\cdot w}\right\}}{w^{\frac{1}{2}}\cdot(1-w)^{\frac{3}{2}}}\cdot dw,$$

where

(A-4) $$-\frac{\mu}{\sqrt{2\cdot\pi}}\cdot u\cdot\int_{y=0}^{u}\frac{\exp\left\{-\frac{\mu^2}{2\cdot y}\right\}}{\left[y\cdot(1-y)\right]^{\frac{3}{2}}}\cdot dy=$$
$$=2\cdot u\cdot\left(1-\mu^2\right)\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\mu\cdot\sqrt{\frac{1-u}{u}}\right)-2\cdot\frac{\mu}{\sqrt{2\cdot\pi}}\cdot u\cdot\sqrt{\frac{u}{1-u}}\cdot\exp\left\{-\frac{\mu^2}{2\cdot u}\right\}$$

and moreover

(A-5) $$\frac{\mu}{\sqrt{2\cdot\pi}}\cdot\int_{w=0}^{u}\frac{\exp\left\{-\frac{\mu^2}{2\cdot w}\right\}}{w^{\frac{1}{2}}\cdot(1-w)^{\frac{3}{2}}}\cdot dw=$$
$$=2\cdot\mu^2\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\mu\cdot\sqrt{\frac{1-u}{u}}\right)+2\cdot\frac{\mu}{\sqrt{2\cdot\pi}}\cdot\sqrt{\frac{u}{1-u}}\cdot\exp\left\{-\frac{\mu^2}{2\cdot u}\right\}.$$

Eq. (A-5) can be easily proved by differentiation.

Furthermore we present the computation of the more complicated integrals

$$2\cdot\sqrt{2\cdot\pi}\cdot\int_{w=0}^{u}\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-w\right)\cdot dx\cdot dw-$$
$$-2\cdot\sqrt{2\cdot\pi}\cdot\int_{w=0}^{u}\int_{x=-\infty}^{0}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x;1-w\right)\cdot dx\cdot dw\,.$$

We shall define

(A-6) $$J\left(u,\kappa;\mu\right):=2\cdot\sqrt{2\cdot\pi}\cdot\int_{w=0}^{u}\int_{x=-\infty}^{\kappa}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-w\right)\cdot dx\cdot dw=$$
$$=2\cdot\sqrt{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\int_{w=0}^{u}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-w\right)\cdot dw\cdot dx$$

and obtain

$$\int_{w=0}^{u}\frac{\partial}{\partial\xi}f\left(x+\mu;w\right)\cdot\frac{\partial}{\partial\xi}f\left(x-2\cdot\kappa;1-w\right)\cdot dw=$$
$$=\int_{w=0}^{u}\mathbf{P}\left[\mathrm{T}_{x+\mu}\left(W\right)\in dw\right]\cdot\frac{\mathbf{P}\left[\mathrm{T}_{x-2\cdot\kappa}\left(W\right)\in dh+1-w\right]}{dh}=$$
$$=\frac{1}{dh}\cdot\int_{w=0}^{u}\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1;\mathrm{T}_{x+\mu}\left(W\right)\in dw\right]=$$
$$=\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1;\min_{0\le\tau\le u}W_{\tau}\le x+\mu\right]=$$
$$=\frac{1}{dh}\cdot\left(\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1\right]-\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1;\min_{0\le\tau\le u}W_{\tau}>x+\mu\right]\right)=$$
$$=\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1\right]-\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1;W_u\in dy;\min_{0\le\tau\le u}W_{\tau}>x+\mu\right]=$$
$$=\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot x+\mu-2\cdot\kappa}\left(W\right)\in dh+1\right]-$$
$$-\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu-y}\left(W\right)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy;\min_{0\le\tau\le u}W_{\tau}>x+\mu\right]=$$
$$=\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu}\left(W\right)\in dh+1\right]-$$
$$-\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu-y}\left(W\right)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy\right]+$$
$$+\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu-y}\left(W\right)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy-2\cdot\left(x+\mu\right)\right]=$$
$$=\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu}\left(W\right)\in dh+1\right]-$$
$$-\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot\left(x-\kappa\right)+\mu-y}\left(W\right)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy\right]+$$

$$+\frac{1}{dh}\cdot\int_{y=-(x+\mu)}^{\infty}\mathbf{P}\left[\mathrm{T}_{-2\cdot\kappa-\mu-y}(W)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy\right].$$

Setting $\lambda(x):=2\cdot x+(\mu-2\cdot\kappa)$ we compute the integral

$$\frac{1}{dh}\cdot\int_{y=x+\mu}^{\infty}\mathbf{P}\left[\mathrm{T}_{2\cdot(x-\kappa)+\mu-y}(W)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy\right]=$$

$$=\frac{1}{2\cdot\pi\cdot(1-u)}\cdot\int_{y=x+\mu}^{\infty}\frac{y-\lambda(x)}{\sqrt{1-u}}\cdot\exp\left\{-\frac{(y-\lambda(x))^2}{2\cdot(1-u)}\right\}\cdot\frac{1}{\sqrt{u}}\cdot\exp\left\{-\frac{y^2}{2\cdot u}\right\}\cdot dy=$$

$$=\frac{\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{2\cdot\pi\cdot(1-u)}\cdot\int_{y=x+\mu}^{\infty}\frac{y-\lambda(x)}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(y-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dy=$$

$$=\frac{\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{2\cdot\pi\cdot(1-u)}\cdot\int_{y=x+\mu}^{\infty}\frac{y-\lambda(x)\cdot u}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(y-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dy-$$

$$-\frac{\lambda(x)\cdot\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{2\cdot\pi}\cdot\int_{y=x+\mu}^{\infty}\frac{1}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(y-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dy=$$

$$=\frac{\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{2\cdot\pi}\cdot\sqrt{\frac{u}{1-u}}\cdot\exp\left\{-\frac{(x+\mu-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}-$$

$$-\frac{\lambda(x)\cdot\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{\sqrt{2\cdot\pi}}\cdot\mathcal{N}\left(\frac{\lambda(x)\cdot u-x-\mu}{\sqrt{u\cdot(1-u)}}\right).$$

We continue with the explicit calculation of, setting $\gamma:=2\cdot\kappa+\mu$.

$$\frac{1}{dh}\cdot\int_{y=-(x+\mu)}^{\infty}\mathbf{P}\left[\mathrm{T}_{-\gamma-y}(W)\in dh+1-u\right]\cdot\mathbf{P}\left[W_u\in dy\right]=$$

$$=\frac{1}{2\cdot\pi\cdot(1-u)}\cdot\int_{y=-(x+\mu)}^{\infty}\frac{\gamma+y}{\sqrt{1-u}}\cdot\exp\left\{-\frac{(\gamma+y)^2}{2\cdot(1-u)}\right\}\cdot\frac{1}{\sqrt{u}}\cdot\exp\left\{-\frac{y^2}{2\cdot u}\right\}\cdot dy=$$

$$=\frac{\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi\cdot(1-u)}\cdot\int_{y=-(x+\mu)}^{\infty}\frac{\gamma\cdot u+y}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(\gamma\cdot u+y)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dy+$$

$$+\frac{\gamma\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi}\cdot\int_{y=-(x+\mu)}^{\infty}\frac{1}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(\gamma\cdot u+y)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dy=$$

$$=\frac{\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi}\cdot\sqrt{\frac{u}{1-u}}\cdot\exp\left\{-\frac{(x+\mu-\gamma\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}+\frac{\gamma\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}}{\sqrt{2\cdot\pi}}\cdot\mathcal{N}\left(\frac{x+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\right).$$

Finally, since $x\le\kappa$ it is a trivial fact that

$$\frac{1}{dh}\cdot\mathbf{P}\left[\mathrm{T}_{2\cdot(x-\kappa)+\mu}(W)\in dh+1\right]=\frac{2\cdot(\kappa-x)-\mu}{\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{(2\cdot(x-\kappa)+\mu)^2}{2}\right\}.$$

In a further step we compute the integrals of those three results along x from $-\infty$ to κ.

$$\int_{x=-\infty}^{\kappa}\frac{\exp\left\{-\frac{\lambda^2(x)}{2}\right\}}{2\cdot\pi}\cdot\sqrt{\frac{u}{1-u}}\cdot\exp\left\{-\frac{(x+\mu-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx+$$

$$+\int_{x=-\infty}^{\kappa}\left(-\lambda(x)\cdot\exp\left\{-\frac{\lambda^2(x)}{2}\right\}\right)\cdot\frac{1}{\sqrt{2\cdot\pi}}\cdot\mathcal{N}\left(\frac{\lambda(x)\cdot u-x-\mu}{\sqrt{u\cdot(1-u)}}\right)\cdot dx.$$

By integration by parts of the second integral and paying attention to the fact that $\frac{d}{dx}\lambda(x)=2$ we obtain

$$\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\mu\cdot u-(\kappa+\mu)}{\sqrt{u\cdot(1-u)}}\right)+$$

$$+\frac{1}{2\cdot\sqrt{2\cdot\pi}}\int_{x=-\infty}^{\kappa}\frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{\lambda^2(x)}{2}\right\}\cdot\exp\left\{-\frac{(x+\mu-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx=$$

$$=\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\mu\cdot u-(\kappa+\mu)}{\sqrt{u\cdot(1-u)}}\right)+$$

$$+\frac{1}{2\cdot\sqrt{2\cdot\pi}}\int_{x=-\infty}^{\kappa}\frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{\lambda^2(x)}{2}-\frac{(x+\mu-\lambda(x)\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx.$$

Now, we have to transform the exponent of the integrand,

$$-\frac{1}{2\cdot u\cdot(1-u)}\cdot\left[(x+\mu-\lambda(x)\cdot u)^2+\lambda^2(x)\cdot u\cdot(1-u)\right].$$

Evidently, this yields

$$\left(x+\mu-\lambda(x)\cdot u\right)^2+\lambda^2(x)\cdot u\cdot(1-u)=$$
$$=(x+\mu)^2-2\cdot(x+\mu)\cdot\lambda(x)\cdot u+\lambda^2(x)\cdot u^2+\lambda^2(x)\cdot u-\lambda^2(x)\cdot u^2\overset{\lambda(x)=2\cdot x+\mu-2\cdot\kappa}{=}$$
$$=(x+\mu)^2-(\mu+2\cdot\kappa)\cdot\lambda(x)\cdot u=(x+\mu)^2-2\cdot x\cdot(\mu+2\cdot\kappa)\cdot u-\left(\mu^2-4\cdot\kappa^2\right)\cdot u=$$
$$=x^2+2\cdot x\cdot\left(\mu\cdot(1-u)-2\cdot\kappa\cdot u\right)+\mu^2\cdot(1-u)+4\cdot\kappa^2\cdot u=$$
$$=\left(x+\mu\cdot(1-u)-2\cdot\kappa\cdot u\right)^2+(\mu+2\cdot\kappa)^2\cdot u\cdot(1-u).$$

Thus we obtain

$$\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\mu\cdot u-(\kappa+\mu)}{\sqrt{u\cdot(1-u)}}\right)+$$
$$+\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{(\mu+2\cdot\kappa)^2}{2}\right\}\cdot\int_{x=-\infty}^{\kappa}\frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{\left(x+\mu\cdot(1-u)-2\cdot\kappa\cdot u\right)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx=$$
$$=\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\mu\cdot u-(\kappa+\mu)}{\sqrt{u\cdot(1-u)}}\right)+$$
$$+\frac{1}{2\cdot\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{(\mu+2\cdot\kappa)^2}{2}\right\}\cdot\mathcal{N}\left((2\cdot\kappa+\mu)\cdot\sqrt{\frac{1-u}{u}}-\frac{\kappa}{\sqrt{u\cdot(1-u)}}\right).$$

We continue with the computation of

$$\frac{\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi}\cdot\sqrt{\frac{u}{1-u}}\cdot\int_{x=-\infty}^{\kappa}\exp\left\{-\frac{(x+\mu-\gamma\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx+\frac{\gamma\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}}{\sqrt{2\cdot\pi}}\cdot\int_{x=-\infty}^{\kappa}\mathcal{N}\left(\frac{x+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\right)\cdot dx=$$
$$=\frac{\exp\left\{-\frac{\gamma^2}{2}\right\}\cdot u}{\sqrt{2\cdot\pi}}\cdot\int_{x=-\infty}^{\kappa}\frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(x+\mu-\gamma\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx+$$
$$+\frac{1}{\sqrt{2\cdot\pi}}\cdot\gamma\cdot(\kappa+\mu-\gamma\cdot u)\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\kappa+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\right)-$$
$$-\frac{\gamma\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi}\cdot\int_{x=-\infty}^{\kappa}\frac{x+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\cdot\exp\left\{-\frac{(x+\mu-\gamma\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}\cdot dx=$$
$$=\frac{1}{\sqrt{2\cdot\pi}}\cdot\left(1-\gamma^2\right)\cdot u\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\kappa+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\right)+$$
$$+\frac{1}{\sqrt{2\cdot\pi}}\cdot\gamma\cdot(\kappa+\mu)\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\kappa+\mu-\gamma\cdot u}{\sqrt{u\cdot(1-u)}}\right)+$$

$$+\frac{\gamma\cdot\exp\left\{-\frac{\gamma^2}{2}\right\}}{2\cdot\pi}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\left\{-\frac{(\kappa+\mu-\gamma\cdot u)^2}{2\cdot u\cdot(1-u)}\right\}.$$

Finally, we compute the last integral

$$\int_{x=-\infty}^{\kappa}\frac{2\cdot(\kappa-x)-\mu}{\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{(2\cdot(x-\kappa)+\mu)^2}{2}\right\}\cdot dx=\frac{1}{\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{\mu^2}{2}\right\}.$$

As a result from all those tedious computations we obtain the following closed formula for $J(u,\kappa;\mu)$,

$$\begin{aligned}\text{(A-7)}\quad J(u,\kappa;\mu)&=2\cdot\exp\left\{-\frac{\mu^2}{2}\right\}-\exp\left\{-\frac{\mu^2}{2}\right\}\cdot\mathcal{N}\left(\frac{\mu\cdot u-(\kappa+\mu)}{\sqrt{u\cdot(1-u)}}\right)-\\&-\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2}\right\}\cdot\mathcal{N}\left((2\cdot\kappa+\mu)\cdot\sqrt{\frac{1-u}{u}}-\frac{\kappa}{\sqrt{u\cdot(1-u)}}\right)+\\&+2\cdot\left(1-(2\cdot\kappa+\mu)^2\right)\cdot u\cdot\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2}\right\}\cdot\mathcal{N}\left(\frac{(\kappa+\mu)-(2\cdot\kappa+\mu)\cdot u}{\sqrt{u\cdot(1-u)}}\right)+\\&+2\cdot(2\cdot\kappa+\mu)\cdot(\kappa+\mu)\cdot\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2}\right\}\cdot\mathcal{N}\left(\frac{(\kappa+\mu)-(2\cdot\kappa+\mu)\cdot u}{\sqrt{u\cdot(1-u)}}\right)+\\&+\frac{2}{\sqrt{2\cdot\pi}}\cdot(2\cdot\kappa+\mu)\cdot\exp\left\{-\frac{(2\cdot\kappa+\mu)^2}{2}\right\}\cdot\sqrt{u\cdot(1-u)}\cdot\exp\left\{-\frac{\left((\kappa+\mu)-(2\cdot\kappa+\mu)\cdot u\right)^2}{2\cdot u\cdot(1-u)}\right\}.\end{aligned}$$

Considering eq. (A-7) for $J(u,0;\mu)$ and comparing this formula to equations (A-4) and (A-5) we obtain the nice relation

$$\text{(A-8)}\quad J(u,0;\mu)=\exp\left\{-\frac{\mu^2}{2}\right\}-\int_{w=0}^{u}\int_{y=0}^{w}\frac{1}{(1-y)^{\frac{3}{2}}}\cdot\mathbf{P}\left[\mathrm{T}_{-\mu}(W)\in dy\right]\cdot dw.$$

APPENDIX B – Some Well-Known Fundamental Results of BROWNian Motion

Definition of a BROWNian Bridge. For the process $Y^{(0)}=\left\{Y_t^{(0)}\right\}_{t=0}^{1}$ with $Y_t^{(0)}=W_t-t\cdot W_1$ we have to prove that for all $\mathrm{C}\in\mathscr{C}[0;1]$ the relation

(B-1) $$\mathbf{P}\left[Y^{(0)}\in \mathrm{C}\right]=\mathbf{P}\left[W\in \mathrm{C}\middle|W_1=0\right]$$

holds. Thus a BROWNian Bridge can be alternatively defined either by the conditional probability $\mathbf{P}\left[W\in \mathrm{C}\middle|W_1=0\right]$ or by the representation as a stochastic process $Y^{(0)}=\left\{Y_t^{(0)}\right\}_{t=0}^{1}$ with $Y_t^{(0)}=W_t-t\cdot W_1$.

PROOF. Evidently, we can assume that $\mathrm{C}\in\mathscr{C}[0;\tau]\subseteq\mathscr{C}[0;1]$ for a certain $0\le\tau\le1$. Then we immediately obtain the following equations using GIRSANOV's theorem and the independence of the increments of a WIENER process.

$$\mathbf{P}\left[Y^{(0)}\in \mathrm{C}\right]=\int_{x\in\mathbb{R}}\mathbf{P}\left[W^{(x)}\in \mathrm{C};W_1\in x+dh\right]=\int_{x\in\mathbb{R}}\mathbf{P}\left[W^{(x)}\in \mathrm{C};W_1^{(x)}\in dh\right]=$$
$$=\int_{x\in\mathbb{R}}\exp\left\{-\frac{x^2}{2}\right\}\cdot\mathbf{P}\left[W\in \mathrm{C};W_1\in dh\right]=\int_{x\in\mathbb{R}}\exp\left\{-\frac{x^2}{2}\right\}\cdot\int_{z\in\mathbb{R}}\mathbf{P}\left[W\in \mathrm{C};W_\tau\in dz;W_1\in dh\right]=$$
$$=\int_{x\in\mathbb{R}}\exp\left\{-\frac{x^2}{2}\right\}\cdot\int_{z\in\mathbb{R}}\mathbf{P}\left[W\in \mathrm{C};W_\tau\in dz\right]\cdot\mathbf{P}\left[W_1-W_\tau\in -z+dh\right]=$$
$$=\frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{x\in\mathbb{R}}\exp\left\{-\frac{x^2}{2}\right\}\cdot\int_{z\in\mathbb{R}}\mathbf{P}\left[W\in \mathrm{C};W_\tau\in dz\right]\cdot\frac{1}{\sqrt{1-\tau}}\cdot\exp\left\{-\frac{z^2}{2\cdot(1-\tau)}\right\}\cdot dh=$$
$$=\int_{z\in\mathbb{R}}\mathbf{P}\left[W\in \mathrm{C};W_\tau\in dz\right]\cdot\frac{1}{\sqrt{1-\tau}}\cdot\exp\left\{-\frac{z^2}{2\cdot(1-\tau)}\right\}=\int_{z\in\mathbb{R}}\mathbf{P}\left[W\in \mathrm{C};W_\tau\in dz\right]\cdot\frac{\mathbf{P}\left[W_1-W_\tau\in -z+dh\right]}{\mathbf{P}\left[W_1\in dh\right]}=$$
$$=\frac{\mathbf{P}\left[W\in \mathrm{C};W_1\in dh\right]}{\mathbf{P}\left[W_1\in dh\right]}=\mathbf{P}\left[W\in \mathrm{C}\middle|W_1=0\right].\square$$

Joint Density of Occupation Times of a WIENER Process.

With respect to the conditional density of occupation times $\Gamma_+(1;0)(W)=\int_{t=0}^{1}\mathbf{1}\left[W_t>0\right]\cdot dt$ of a WIENER process

(B-2) $$\mathbf{P}\left[\Gamma_+(1;0)(W)\in u+dh\middle|W_1=0\right]=du\,,\ 0\le u\le1,$$

we shall directly derive the formula for the joint density $\mathbf{P}\left[\Gamma_+(1;0)(W)\in du;W_1\in dx\right]$ for all $0\le u\le1$ and for all $x\in\mathbb{R}\setminus\{0\}$.

For the proof we use the last zero $\Theta_0(W):=\max\left\{0\le t\le1\middle|W_t=0\right\}$ of the WIENER process in the interval $[0;1]$.

PROOF. First we consider the case $x<0$ and derive the following equations.

$$\mathbf{P}\left[\Gamma_+(1;0)(W)\in du; W_1\in dx\right]=\int_{t=u}^{1}\mathbf{P}\left[\Gamma_+(1;0)(W)\in du;\Theta_0(W)\in dt; W_1\in dx\right]=$$

$$=\int_{t=u}^{1}\mathbf{P}\left[\Gamma_+(t;0)(W)\in du; W_t=0\right]\cdot\mathbf{P}\left[\mathrm{T}_{(-x)}(W)\in 1-t+dh\right]=$$

$$=\int_{t=u}^{1}\mathbf{P}\left[\Gamma_+(1;0)(W)\in\frac{du}{t}; W_1=0\right]\cdot\mathbf{P}\left[\mathrm{T}_{(-x)}(W)\in 1-t+dh\right]=$$

$$=\int_{t=u}^{1}\mathbf{P}\left[\Gamma_+(1;0)(W)\in\frac{du}{t}\middle|W_1=0\right]\cdot\mathbf{P}\left[W_1\in dh\right]\cdot\mathbf{P}\left[\mathrm{T}_{(-x)}(W)\in 1-t+dh\right]=$$

$$=\frac{|x|}{2\cdot\pi}\int_{t=u}^{1}\frac{\exp\left\{-\frac{x^2}{2\cdot(1-t)}\right\}}{\left[t\cdot(1-t)\right]^{\frac{3}{2}}}\cdot dt\cdot du\cdot dx\,.$$

By an analogous argument the formula for $x>0$ can be provided,

$$\mathbf{P}\left[\Gamma_+(1;0)(W)\in du; W_1\in dx\right]=\frac{|x|}{2\cdot\pi}\cdot\int_{t=1-u}^{1}\frac{\exp\left\{-\frac{x^2}{2\cdot(1-t)}\right\}}{\left[t\cdot(1-t)\right]^{\frac{3}{2}}}\cdot dt\cdot du\cdot dx\,.$$

Uniform Distribution of a BROWNian Bridge

According to BILLINGSLEY's remark [cf. BILLINGSLEY (1968), eq. (11.42)] the uniform distribution (B-2) can be elementarily proved by considering a random walk process $S=\{S_n\}_{n\in\mathbb{N}_0}$ with $S_n:=\sum_{i=1}^{n}\xi_i$, where $\{\xi_i\}_{i\in\mathbb{N}}$ is a sequence of independent, identically distributed random variables with $\mathbf{P}\left[\xi_i=\pm 1\right]=\frac{1}{2}$ for all $i\in\mathbb{N}$. Furthermore, the random variable $V=\#\left\{1\le k\le 2\cdot m\middle|S_{k-1}\ge 0\wedge S_k\ge 0\right\}$ is introduced. Then, we shall prove that the following equation holds for all $0\le i\le m$,

$$\mathbf{P}\left[V=2\cdot i\middle|S_{2\cdot m}=0\right]=\frac{1}{m+1}\,.$$

PROOF. For $m=0$ the equation trivially holds. Thus we only have to consider values $m\in\mathbb{N}$. We set $\Upsilon_i:=\left[V=2\cdot i\middle|S_{2\cdot m}=0\right]$ for $0\le i\le m$. In a first step it is proved that

$S_{2\cdot k}(\omega)=0$

$$\mathbf{P}\left[V=0\middle|S_{2\cdot m}=0\right]=\mathbf{P}\left[V=2\middle|S_{2\cdot m}=0\right].$$

Let now be $\Theta_0(S)$ the last zero of S prior to $2\cdot m$, i.e.

$$\Theta_0(S):=\max\left\{0\le 2\cdot k<2\cdot m\middle|S_{2\cdot k}=0\right\}.$$

Since $\forall \omega \in \Upsilon_0 : \exists 0 \le k < m : \Theta_0\left(S(\omega)\right) = 2 \cdot k$, we obtain a partition of Υ_0 into m pairwise disjoint classes $\Upsilon_0^{(k)}$, $0 \le k \le m-1$. Moreover, a similar partition of Υ_1 into m pairwise disjoint classes $\Upsilon_1^{(k)}$, $0 \le k \le m-1$, with $\Upsilon_1^{(k)} = \left\{\omega \in \Upsilon_1 \left| S_{2\cdot k+1} = 1\right.\right\}$. Then using the one-to-one correspondence of ω and $S(\omega)$ the mapping $\Lambda_0^{(k)} : \Upsilon_0^{(k)} \to \Upsilon_1^{(k)}$ with $\Lambda_0^{(k)}\left(S(\omega)\right)_j = S_j(\omega)$ for all $0 \le j \le 2 \cdot k$, $\Lambda_0^{(k)}\left(S(\omega)\right)_{2\cdot k+1} = 1$ and $\Lambda_0^{(k)}\left(S(\omega)\right)_j = S_{j-1}(\omega) + 1$ for all $2 \cdot k + 2 \le j \le 2 \cdot m$ is a one-to-one mapping. Evidently, the mapping $\Lambda_0 : \Upsilon_0 \to \Upsilon_1$ with $\Lambda_0\left(S(\omega)\right) = \Lambda_0^{(k)}\left(S(\omega)\right)$ for all $\omega \in \Upsilon_0^{(k)}$ is a one-to-one mapping, too, and the equation $\mathbf{P}(\Upsilon_0) = \mathbf{P}(\Upsilon_1)$ is proved. In the next step we have to prove the general case $\mathbf{P}(\Upsilon_i) = \mathbf{P}(\Upsilon_{i+1})$ for $1 \le i < m$. First we introduce a last zero passage point functional

$$\Xi_0(S) := \max\left\{0 \le 2 \cdot j \le 2 \cdot m \left| \left(j > 0 \Rightarrow S_{2\cdot j-1} > 0\right) \wedge S_{2\cdot j} = 0 \wedge \left(j < m \Rightarrow S_{2\cdot j+1} < 0\right)\right.\right\},$$

furthermore for $\Xi_0(S) > 0$ the zero passage point functional

$$Z_0(S) := \max\left\{0 \le 2 \cdot j < \Xi_0(S) \left| \left(j > 0 \Rightarrow S_{2\cdot j-1} < 0\right) \wedge S_{2\cdot j} = 0 \wedge S_{2\cdot j+1} > 0\right.\right\}$$

and the last zero $\Theta_0^{\Xi_0}(S)$ prior to $\Xi_0(S) > 0$. Evidently, $Z_0(S) \le \Theta_0^{\Xi_0}(S)$ holds. Now Υ_{i+1} can be decomposed by the partition $\Upsilon_{i+1}^{(k)} := \left\{\omega \in \Upsilon_{i+1} \left| \Xi_0\left(S(\omega)\right) = 2 \cdot k\right.\right\}$ with $\Upsilon_{i+1} = \sum_{k=1}^{m} \Upsilon_{i+1}^{(k)}$. Since for all $\omega \in \Upsilon_{i+1}$ there exists a unique $1 \le k \le m$ such that $\omega \in \Upsilon_{i+1}^{(k)}$ we define the mapping $\left(\Lambda_i^{(k)}\right)^{-1} : \Upsilon_{i+1}^{(k)} \to \Upsilon_i$. For fixed $\omega \in \Upsilon_{i+1}^{(k)}$ we then have $Z_0\left(S(\omega)\right) = 2 \cdot j$, $\Theta_0^{\Xi_0}\left(S(\omega)\right) = 2 \cdot h$ and $\Xi_0\left(S(\omega)\right) = 2 \cdot k$ with $2 \cdot j \le 2 \cdot h < 2 \cdot k$ such that

$$S_{2\cdot j}(\omega) = S_{2\cdot h}(\omega) = S_{2\cdot k}(\omega) = S_{2\cdot m}(\omega) = 0,$$

$S_\nu(\omega) \ge 0$ for all $2 \cdot j < \nu < 2 \cdot h$, $S_\nu(\omega) > 0$ for all $2 \cdot h < \nu < 2 \cdot k$ and $S_\nu(\omega) \le 0$ for all $2 \cdot k < \nu < 2 \cdot m$. Since $2 \cdot j \le 2 \cdot h$ we set $\left(\Lambda_i^{(k)}\right)^{-1}\left(S(\omega)\right)_\nu = S_\nu(\omega)$ for all $0 \le \nu \le 2 \cdot j$, $\left(\Lambda_i^{(k)}\right)^{-1}\left(S(\omega)\right)_{2\cdot j+\nu} = S_{2\cdot h+1+\nu}(\omega) - 1$ for all $0 \le \nu \le 2 \cdot (m-h) - 1$ and finally $\left(\Lambda_i^{(k)}\right)^{-1}\left(S(\omega)\right)_{2\cdot(m+j-h)+\nu} = S_{2\cdot j+\nu}(\omega)$ for all $0 \le \nu \le 2 \cdot (h-j)$. Then the mapping $\left(\Lambda_i\right)^{-1} : \Upsilon_{i+1} \to \Upsilon_i$ with $\left(\Lambda_i\right)^{-1}\left(S(\omega)\right) := \left(\Lambda_i^{(k)}\right)^{-1}\left(S(\omega)\right)$ is a one-to-one mapping, as can be seen from definition, and so is $\Lambda_i : \Upsilon_i \to \Upsilon_{i+1}$. Thus we have $\mathbf{P}(\Upsilon_i) = \mathbf{P}(\Upsilon_\iota)$ for all $0 \le i, \iota \le m$, and since $\Omega = \sum_{i=0}^{m} \Upsilon_i$ the assertion holds.□

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. New York: Wiley.
[2] PECHTL, A. (1999) "Distributions of Occupation Times of Brownian Motion with Drift." *Journal of Applied Mathematics & Decision Sciences*, **3**(1), 41–62.