Local Expected Shortfall-Hedging

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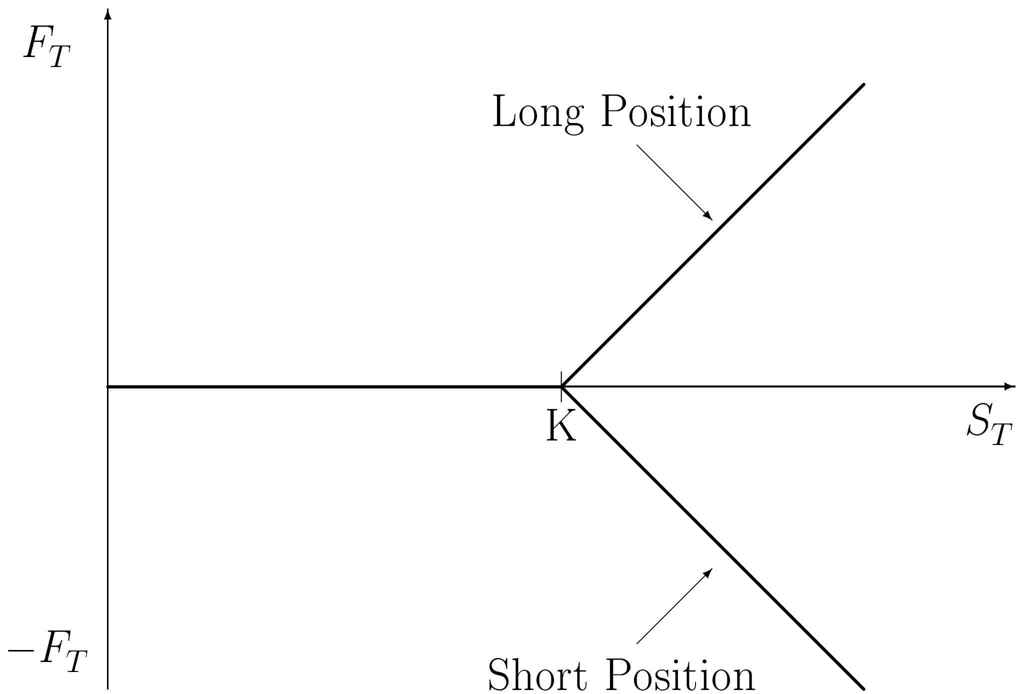
Outline:

1. Introduction
2. Model Framework
3. Hedging Approches
4. Expected Shortfall-Hedging
5. Local Expected Shortfall-Hedging
6. Conclusions

1. Introduction

- Let us assume a situation where an investor has written a European call option on a stock, say for a price of 10 €.

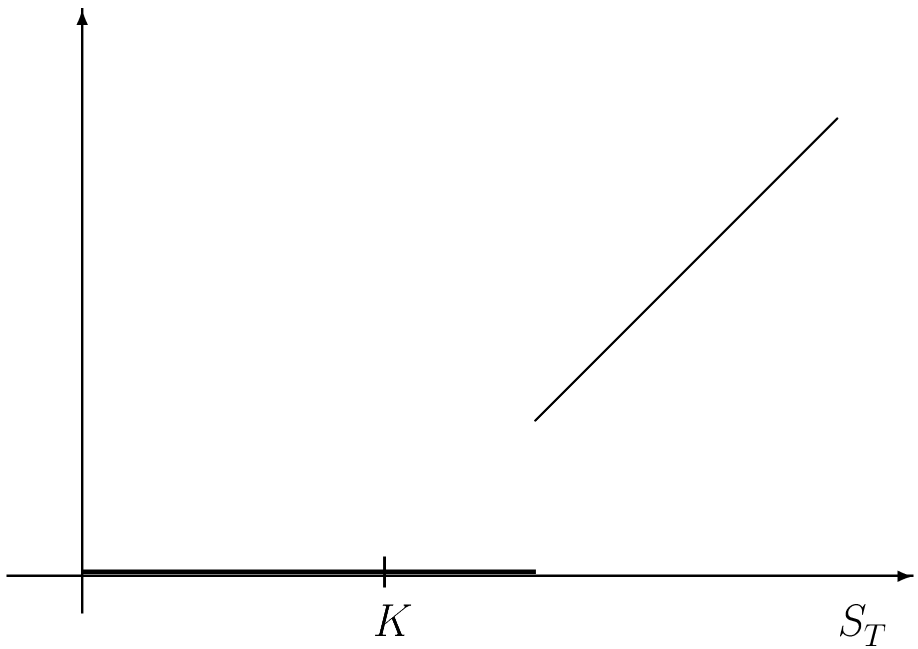


- Suppose that the market is incomplete and that the investor is unwilling to follow a superhedging strategy which requires very often to buy one unit of the underlying instrument for, say 100 **€**.

- **Question: What is the optimal self-financing hedging strategy under a constraint on the initial hedging capital?**

- Föllmer/Leukert (2000) propose a self-financing hedging strategy which minimizes the **expected shortfall** in a Black/Scholes (1973) model. This approach is in the spirit of the martingale approach of portfolio optimization.

- **One answer:** Minimizing the investor's expected shortfall under a budget constraint is tantamount to (super-) hedge a suitable gap option ("modified claim").



- **Question:** Why should an investor not follow a replication strategy if the market is frictionless and complete? (Risk-averse investors would generally follow a perfect hedging strategy in a complete markets setting like the one assumed by Föllmer/Leukert (2000)).

2. Model Framework

We assume a situation where an investor has written a European contingent claim on a stock and wants to hedge the occuring risk with a fixed but arbitrary initial hedging capital $\bar{V}_0$.

- **Hedging Object:** Short position of a European contingent claim F_T.

- **Hedging Instruments:**
  - Underlying stock $S = (S_0, S_1, \ldots, S_T)$ and
  - riskless money market account $B_t = (1+r)^t$, $t = 0, 1, \ldots, T$.

- **Hedging Strategies:** To hedge the contingent claim the investor chooses a strategy $H = (h, h^0)$ where $h_t(h_t^0)$ represents the quantity of the stock (money market account) held in the portfolio at time t.

  The value of a hedging strategy is $V_t(H) = h_t \cdot S_t + h_t^0 \cdot B_t$.

  The set of all self-financing strategies is denoted by $\mathcal{H}_{\mathbf{S}}$.

3. Hedging Approaches

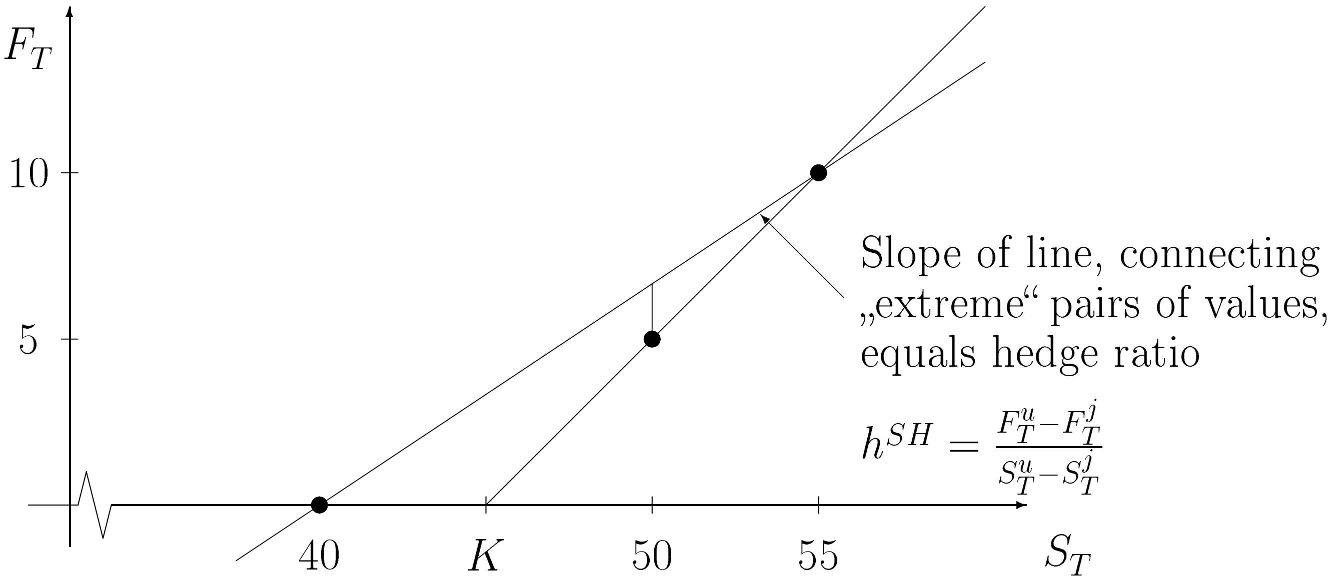
<table>
<tr><td></td><td>Complete Markets</td><td>Incomplete Markets</td><td></td></tr>
<tr><td>No Shortfall Risk</td><td>Delta-Hedging:
Black/Merton/Scholes (1973)
Cox/Ross/Rubinstein (1979)</td><td>Superhedging:
El Karoui/Quenez (1995)
Naik/Uppal (1992)</td><td rowspan="2">No Restriction on Initial Hedging Capital</td></tr>
<tr><td rowspan="3">Shortfall Risk</td><td></td><td>Local Risk-Hedging:
Föllmer/Schweizer (1991)
Schweizer (1992)</td></tr>
<tr><td colspan="2">Global Variance-Hedging: Schweizer (1996)</td><td>————————</td></tr>
<tr><td colspan="2">Shortfall Probability-Hedging:
Föllmer/Leukert (1999)
Global Expected Shortfall-Hedging:
Föllmer/Leukert (2000), Cvitanić/Karatzas (1999)
Cvitanić (1998), Schulmerich/Trautmann (2001),
Schulmerich(2001)
Local Expected Shortfall-Hedging:
Schulmerich/Trautmann (2001), Schulmerich(2001)</td><td>Restriction on Initial Hedging Capital</td></tr>
</table>

3.1 No Shortfall-Risk in Incomplete Markets: Superhedging

A strategy $H \in \mathcal{H}$ is called a superhedging strategy if $V_T(H) \geq F_T$ P-a.s.
The time t cost of carrying out the cheapest superhedging strategy is given by the supremum of the expected terminal value over all Equivalent Martingale Measures $Q \in \mathcal{Q}$:

$$\inf_{H \in \mathcal{H}} \{V_t(H) \mid V_T(H) \geq F_T \text{ P-a.s.}\} = \sup_{Q \in \mathcal{Q}} E_Q(F_T \mid \mathcal{F}_t)/B_{T-t}.$$

Example 1 (Superhedging in a trinomial one-period model)



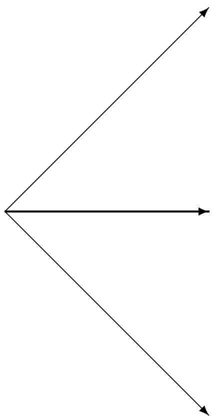
3.2 Local Risk-Hedging in the Trinomial Model

Example 2 (One-period trinomial Model)

Process parameters $U = 1.1$, $D = 1$, $J = 0.8$, interest rate $r = 0\ \%$.

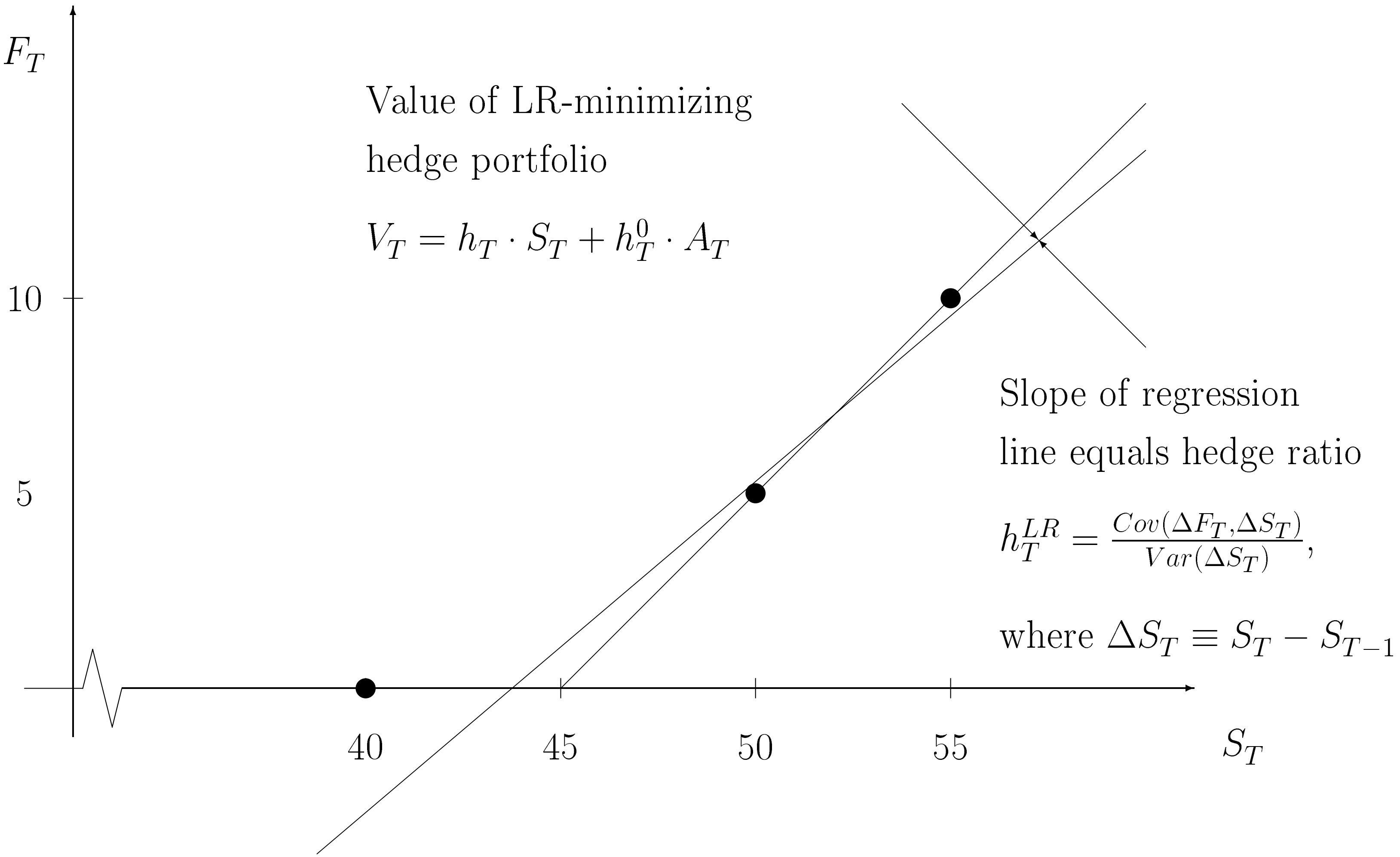
With two traded assets (stock and money market account) we get the following Equivalent Martingale Measures (EMMs):

$S = 50$



Stock	Call	Physical Measure	EMMs
S_T	F_T	P	Q
55	10	0.57	$2q(\omega_J)$
50	5	0.42	$1 - 3q(\omega_J)$
40	0	0.01	$0 < q(\omega_J) < 1/3$

Local Risk-Hedging in the Trinomial Model: A graphical illustration



3.3 Shortfall-Based Hedging Approaches

Motivation:

- In complete markets: hedger is not willing to invest completely the proceeds from writing the option.
- In incomplete markets: hedger is not willing or able to finance a superhedging strategy.

Measures of Shortfall Risk:

- Shortfall Probability (not a coherent risk measure, → Quantile Hedging)
- Expected Shortfall (no coherent risk measure)

Two-Step Procedure

In accordance with the *martingale approach to portfolio optimization* the following two-step procedure is suitable:

(1) Calculation of a modified contingent claim (MCC) X which is attainable given that the initial hedging capital $\bar{V}_0$ is less than F_0.

(2) Superhedging of the modified contingent claim X

4. Expected Shortfall-Hedging

4.1 (Discounted) Expected Shortfall is a quasi-coherent Risk-Measure

Expected Shortfall of a risky position X is defined through

$$\rho(X) = ESD(X) = E_P(\max(-X/B_T; 0)) \equiv E_P(X^-/B_T)$$

fulfills:

Axiom S: *(Subadditivity)* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Axiom PH: *(Positive homogeneity)* $\rho(\alpha \cdot X) = \alpha \cdot \rho(X)$ when $\alpha \geq 0$.

Axiom M: *(Monotonicity)* $\rho(Y) \leq \rho(X)$ when $X \leq Y$.

but not

Axiom T: *(Translation invariance)* $\rho(X + \alpha \cdot B_T) = \rho(X) - \alpha$.

$\rho(x) = ESD(X)$ fulfills instead

Axiom T': For all risky positions X and all real numbers α we have the inequality

$$-B_T^{-1} \cdot E_P(X + \alpha \cdot B_T) \leq \rho(X) - \alpha. \qquad \alpha \in IR$$

4.2 Problem and two-step solution

- **Problem ES:** Find a self-financing strategy H which minimizes

$$E_P[(F_T - V_T(H))^+] \text{ under the constraints } V_0(H) = \bar{V}_0 \text{ and } H \in \mathcal{H}_S.$$

- **Solution:** Föllmer/Leukert (2000), Cvitanić/Karatzas (1999), and Pham (1999) propose a two-step procedure similar to the martingale approach of portfolio optimization:

Step 1: **Static optimization problem:** (easy to solve if $\mathcal{Q}$ is a singleton)

$$\max_X E_P(X) \text{ under the constraints } \sup_{Q\in\mathcal{Q}} E_Q(X/B_T) \leq \bar{V}_0 \text{ and } F_T \geq X.$$

Step 2: **Representation problem:**

Superhedge the modified claim X^* calculated in step 1.

4.3 Expected Shortfall-Hedging

Proposition (ES-Hedging without a shortfall bound) *In a complete market the optimal modified contingent claim X^* has the representation*

$$X^*(\omega) = F_T(\omega) 1_{\{\frac{P}{Q}(\omega) > c_{ES}\}} + \gamma 1_{\{\frac{P}{Q}(\omega) = c_{ES}\}}$$

where

$$c_{ES} = \min_{\omega \in \Omega} \{P(\omega)/Q(\omega)\}$$

and

$$\gamma = (\bar{V}_0 \cdot B_T - E_Q(1_{\{P/Q > c_{ES}\}} F_T))/(E_Q(1_{\{P/Q = c_{ES}\}}))$$

Remarks:

- Replicating the modified contingent claim X_T^* with strategy (V_0^*, H^*) minimizes the expected shortfall under the constraint $V_0 \leq \bar{V}_0$.
- In contrast to Föllmer/Leukert (2000) our approach does not require that $X^* \geq 0$.
  This relaxation results typically in a lower expected shortfall.

The requirement $F_T - X^* \leq b$ leads to the following

Proposition (ES-Hedging with a shortfall bound) *In a complete market the optimal modified contingent claim X^* has the representation*

$$X^*(\omega) = F_T(\omega)1_{\{\frac{P}{Q}(\omega)>c_{ES}\}} + \gamma 1_{\{\frac{P}{Q}(\omega)=c_{ES}\}} + (F_T(\omega) - b)1_{\{\frac{P}{Q}(\omega)<c_{ES}\}}$$

where

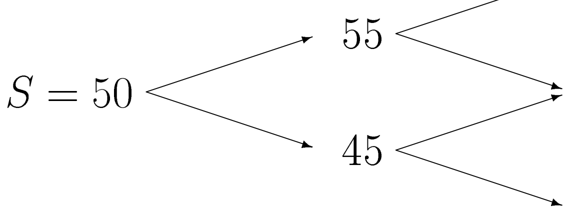
$$c_{ES} = \arg\min_{c \in I\!R_+} \{E_Q(F_T 1_{\{P/Q(\omega)>c\}} + (F_T - b)1_{\{P/Q(\omega)\leq c\}} \leq \bar{V}_0 B_T\}$$

and

$$\gamma = (\bar{V}_0 \cdot B_T - E_Q(F_T 1_{\{P/Q(\omega)>c_{ES}\}} F_T) - E_Q((F_T - b)1_{\{P/Q(\omega)<c_{ES}\}}))/(E_Q(1_{\{P/Q(\omega)=c_{ES}\}}))$$

Example 3 (Expected Shortfall Hedging in the Binomial Model)

Process parameters $U = 1.1$, $D = 0.9$, $p^u = 0.80$, $p^d = 0.20$, interest rate $r = 0$ %, strike price $K = 45$ and initial hedging capital $\bar{V}_0 = 4$.



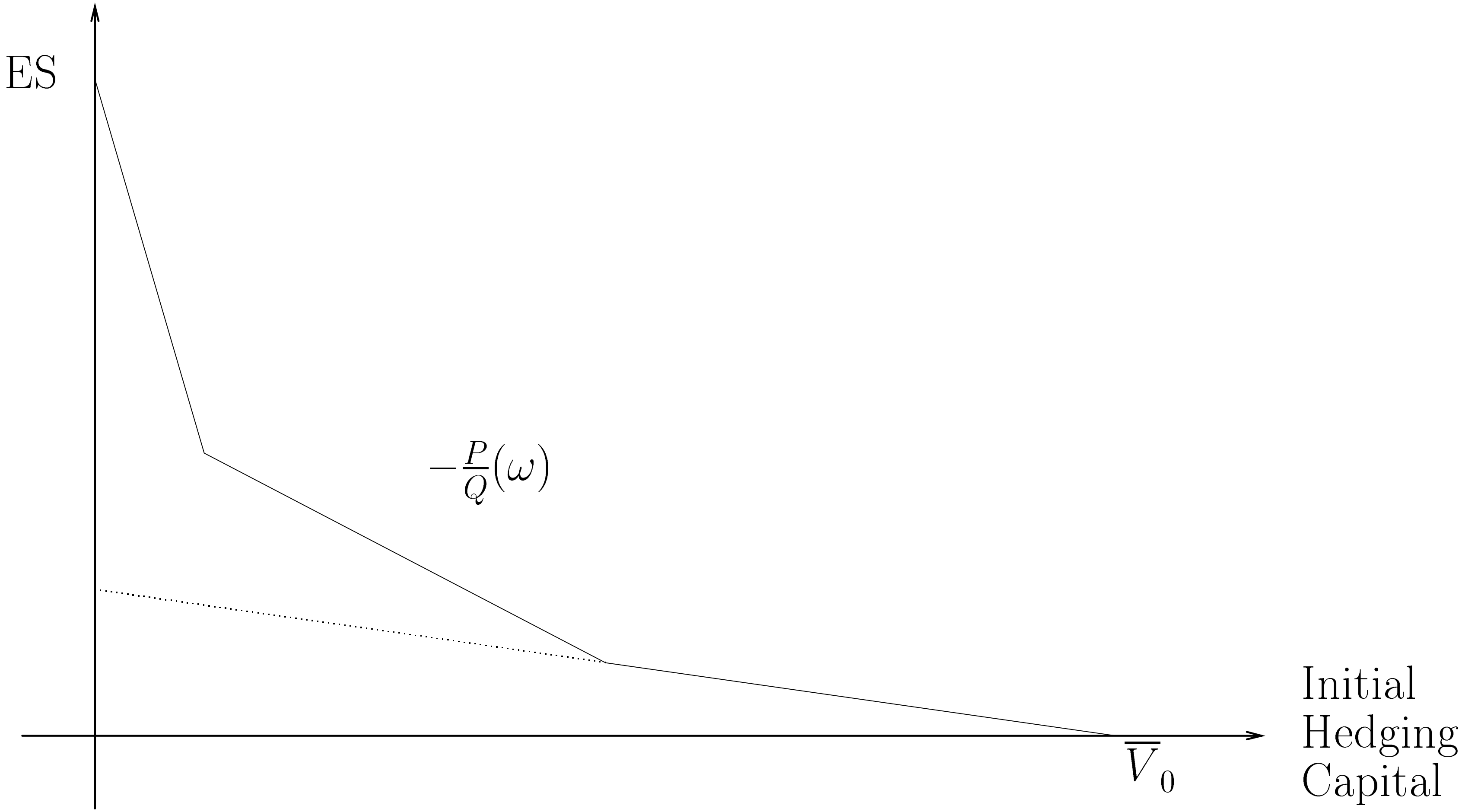
				$V_T(H^*)$ according to	
S_T	F_T	P	Q	F/L	S/T
60.5	15.5	0.6400	0.25	15.5	15.5
49.5	4.5	0.3200	0.5	0.25	4.5
40.5	0	0.0400	0.25	0	-8.5

Expected Shortfall:

ESF(F/L): $0.3364 \cdot 0 + 0.4872 \cdot 4.25 + 0.1764 \cdot 0 = 2.07$ (Föllmer/Leukert, 2000)

ESF(S/T): $0.3364 \cdot 0 + 0.4872 \cdot 0 + 0.1764 \cdot 8.5 = 1.5$ (Schulmerich/Trautmann, 2001)

Expected Shortfall as a function of the Initial Capital



- **Question:** How to solve the static optimization problem if $\mathcal{Q}$ is of infinite size, i.e. if markets are incomplete?

- **Our answer:** SOP-algorithm for solving the static optimization problem in a discrete model

(SOP0) *Initialization*: Set $i \equiv 1$ and define $Q_1 \equiv \arg\max_{Q \in \bar{\mathcal{Q}}} E_Q(F_T)$.

(SOP1) *Iteration*: Maximize $E_P(X^i)$ under the constraints $X^i \leq F_T$ and $\max_{j=1,\ldots,i} E_{Q_j}(X^i/B_T) \leq \bar{V}_0$.

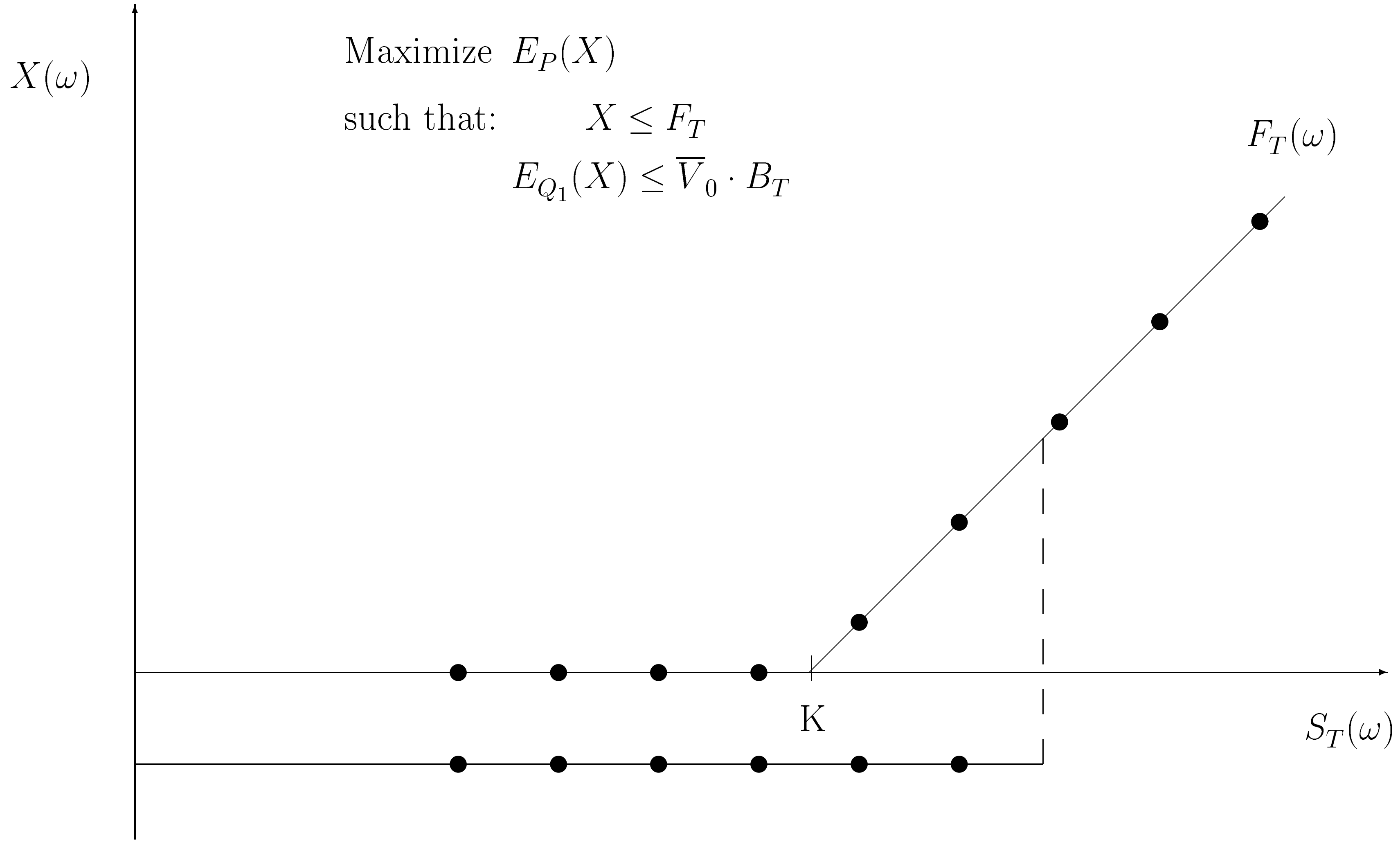
(SOP2) *Termination test*: If the price of the superhedging strategy for X^i obeys the budget constraint, i.e. $\max_{Q \in \bar{\mathcal{Q}}} E_Q(X^i/B_T) \leq \bar{V}_0$, then the modified claim X^i is optimal in the sense of the static optimization problem and the algorithm terminates. Otherwise, define $Q_{i+1} \equiv \arg\max_{Q \in \bar{\mathcal{Q}}} E_Q(X^i/B_T)$, increase $i = i+1$ and return to step (SOP1)

Static Optimization Problem (SOP1)

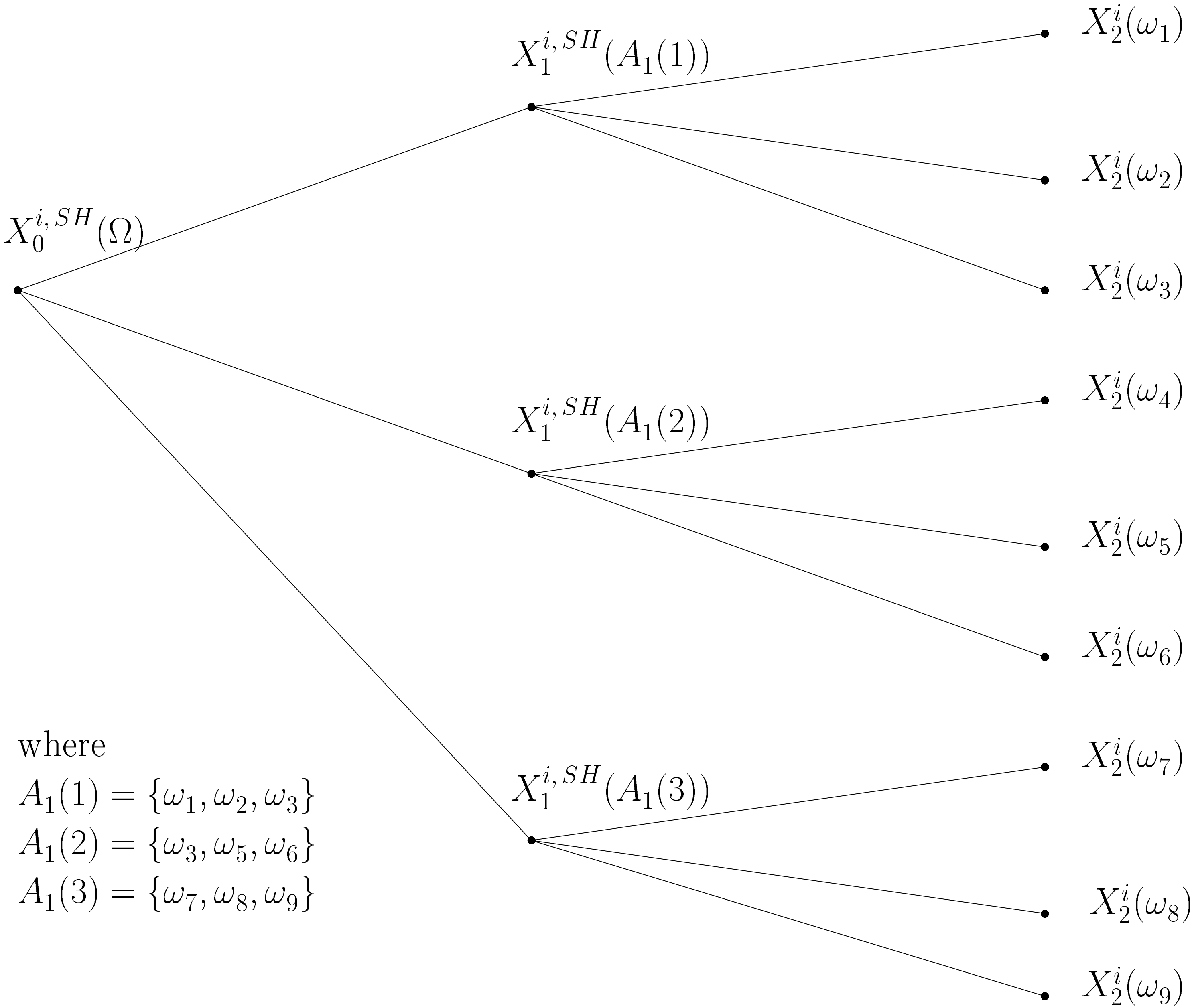
$$\text{Maximize } E_P(X)$$

$$\text{such that: } \quad X \leq F_T$$

$$E_{Q_1}(X) \leq \overline{V}_0 \cdot B_T$$



Recursive Calculation of Superhedging Values (SOP2)



Superhedging requires to solve 4 LPs here and $\sum_{t=0}^{n-1} 3^t$ LPs in general

where

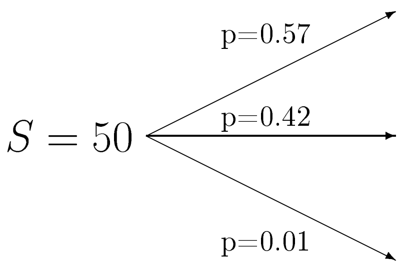
$A_1(1) = \{\omega_1, \omega_2, \omega_3\}$

$A_1(2) = \{\omega_3, \omega_5, \omega_6\}$

$A_1(3) = \{\omega_7, \omega_8, \omega_9\}$

Example 4 (Expected shortfall-hedging in the trinomial model)

Process parameters $U = 1.1$, $D = 1$, $J = 0.8$, strike price $K = 45$, interest rate $r = 0$ %, initial hedging capital $\bar{V}_0 = 5.275$.



S_T	F_T	Terminal value V_T of optimal hedging strategy according to F/L	S/T	LR
55	10	7.91	10	9.86
50	5	5	5	5.275
40	0	0	-4.175	-3.9

Expected Shortfall:

ESF(F/L):	$0.57 \cdot (10 - 7.91) + 0.42 \cdot 0 + 0.01 \cdot 0 = 1,19$	(Föllmer/Leukert, 2000)
ESF(S/T):	$0.57 \cdot 0 + 0.42 \cdot 0 + 0.01 \cdot (0 + 4.175) = 0.04$	(Schulmerich/Trautmann, 2001)
LR:	$0.57 \cdot (10 - 9.86) + 0.42 \cdot 0 + 0.01 \cdot 3.9 = 0.12$	(Schweizer, 1992)

5. Local Expected Shortfall-Hedging

- **Idea:** We partition the complex overall problem ES into several one-period problems and minimize the expected shortfall only locally.

- **Problem LES:**

$$\min_H \sum_{t=1}^{T} E_P[(F_t^{SH} - V_t(H))^+ \mid \mathcal{F}_{t-1}] \text{ under the constraints } V_0(H) = \bar{V}_0 \text{ and } H \in \mathcal{H}_S$$

- **Solution:**

  Calculate ES-strategies in a one-period model n-times via SOP-algorithm!

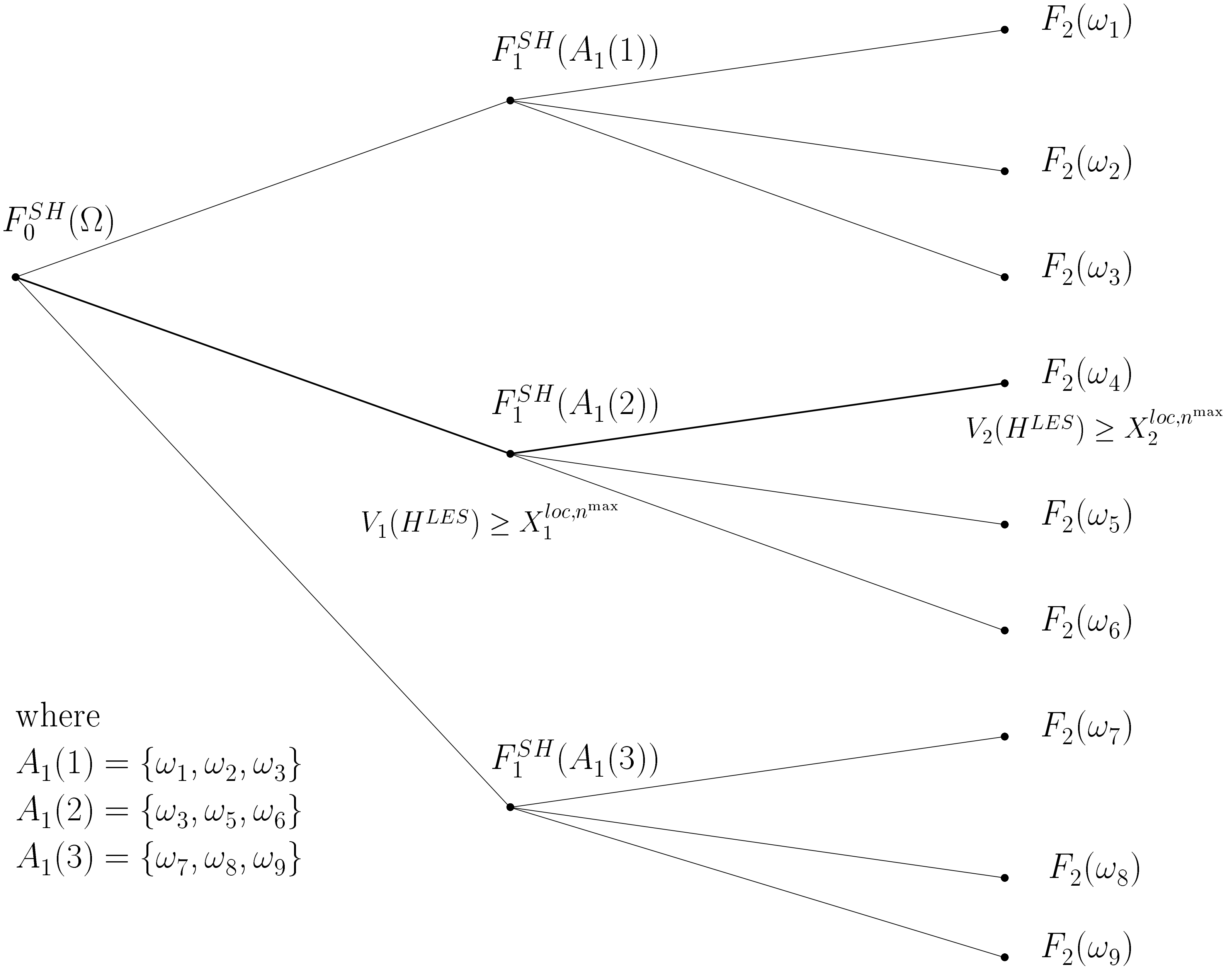
- **Properties:**

  LES- and ES-strategy coincide in the one-period case.

  LES- and ES-strategy coincide in the binomial model.

  LES- and ES-strategy coincide if $\bar{V}_0$ is sufficiently high.

Iterative Calculation of the LES-Strategy



where

$A_1(1) = \{\omega_1, \omega_2, \omega_3\}$

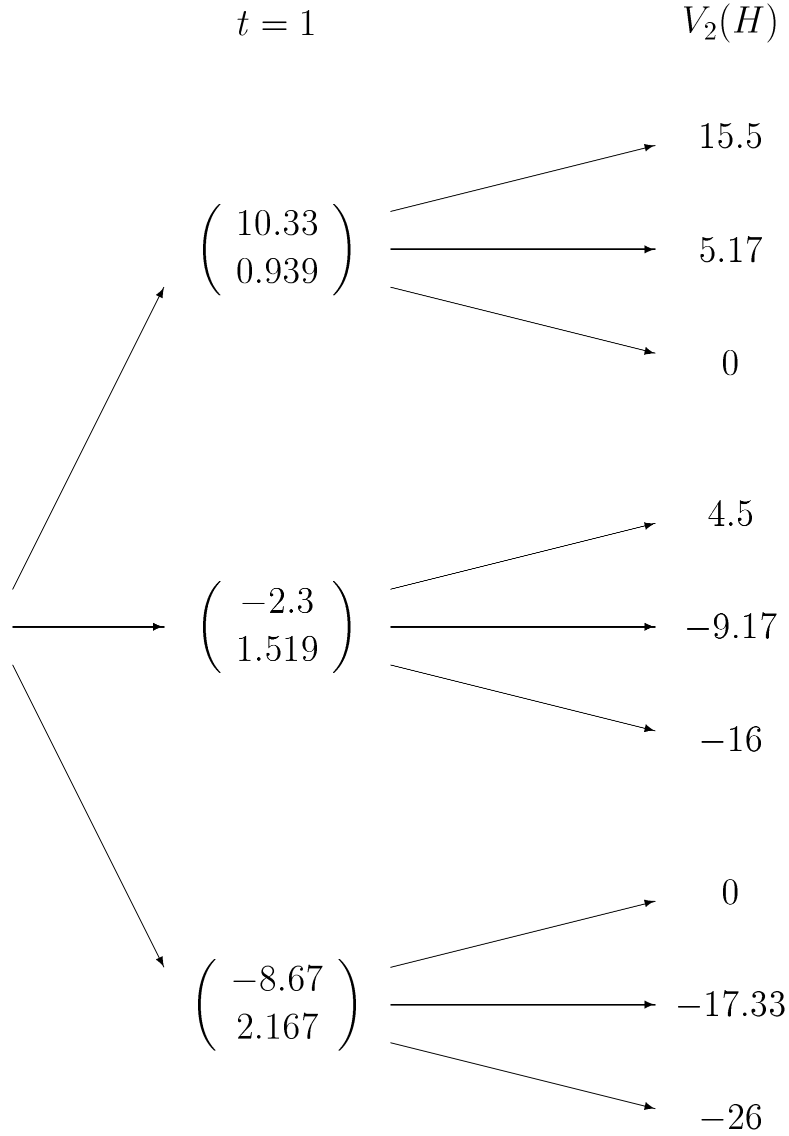
$A_1(2) = \{\omega_3, \omega_5, \omega_6\}$

$A_1(3) = \{\omega_7, \omega_8, \omega_9\}$

Example (LES-strategy without a shortfall bound ($b = \infty$))

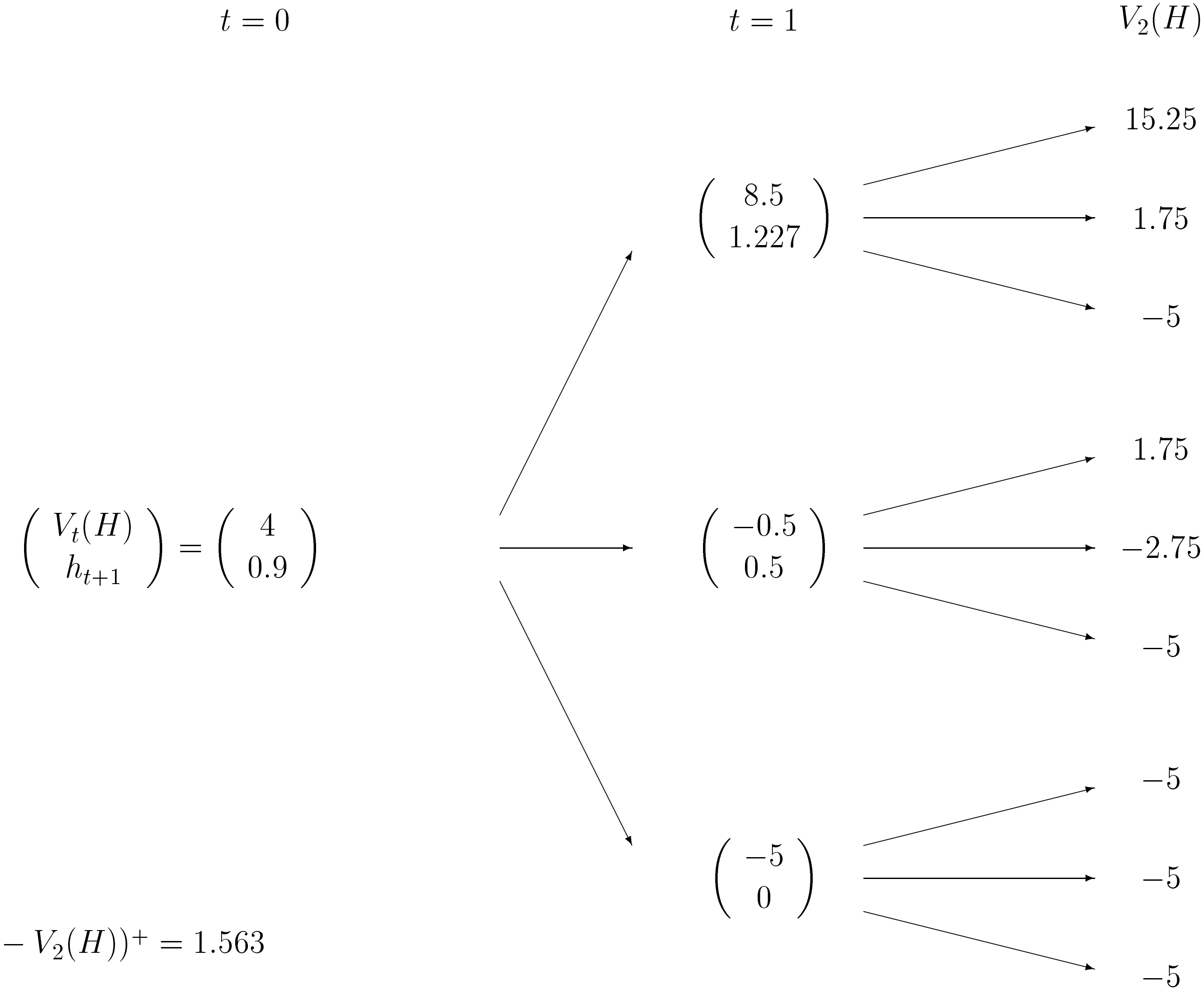
$t = 0$

$$\begin{pmatrix} V_t(H) \\ h_{t+1} \end{pmatrix} = \begin{pmatrix} 4 \\ 1.267 \end{pmatrix}$$



$E_P(F_2 - V_2(H))^+ = 0.765$

Example (LES-strategy with a shortfall bound $b = 5$)

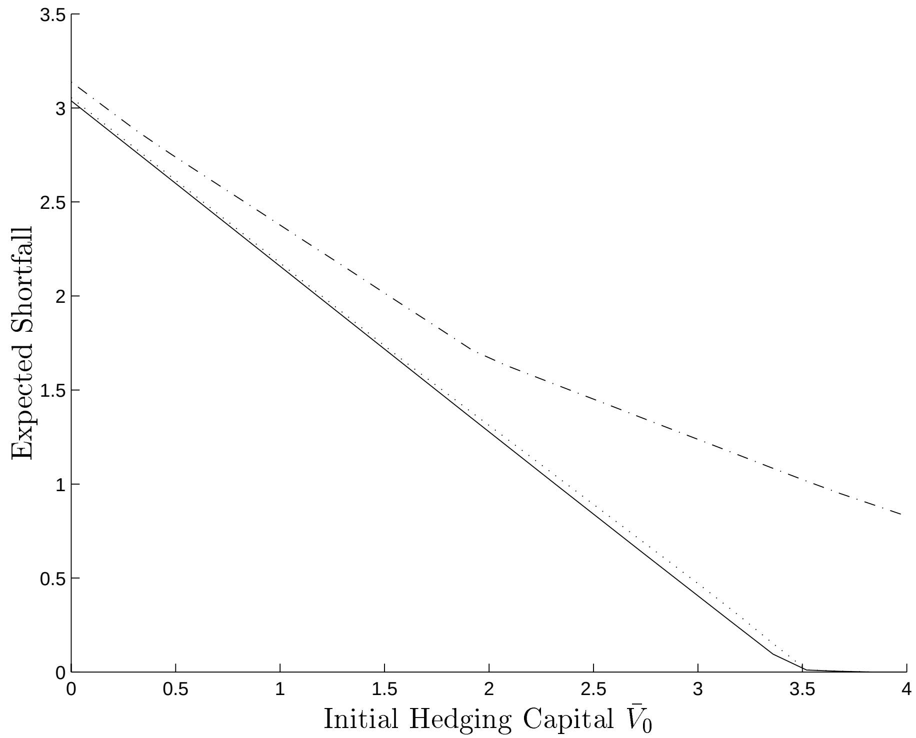


$E_P(F_2 - V_2(H))^+ = 1.563$

The efficient frontier of the ES- and LES-strategy

Expected Shortfall vs. Initial Hedging Capital

Parameter values: initial stock price = 50€; annual interest rate (r) = 5%; annual volatility of the "normal" stock price return (σ) = 20%; annual expected rate of the "normal" return of the stock (α) = 15%; time to maturity of the option (τ) = 1/12; strike price of the option (K) = 47€; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 3.



Computational complexity of ES- and LES-strategies

Panel A

	Main loops	Iterations per main loop (=no. of mcc)	No. of LP's per iteration	Total number of LP'S (incl. number of LP's to be solved for initilization)
ES-strategy	1	$\sim 3^n \cdot (n-1)$ (average value)	$1+\frac{3^n-1}{2}$	$\sim 3^n \cdot (n-1) \cdot (1+\frac{3^n-1}{2}) + \sum_{t=0}^{n-1} \binom{t+2}{2}$ (average value)
LES-strategy	n	≤ 2	$1+\frac{3^1-1}{2}=2$	$\leq 4n + n + \sum_{t=0}^{n-1} \binom{t+2}{2}$

Computational complexity of ES- and LES-strategies

Panel B

Number of constraints in linear programs	Number of Periods $n=2$ ES	$n=2$ LES	$n=3$ ES	$n=3$ LES	$n=4$ ES	$n=4$ LES	$n=5$ ES	$n=5$ LES
1	1	2	1	3	1	4	1	5
2	29	12	443	20	5.881	34	97.406	53
3	1	0	1	0	1	0	1	0
≥ 4	3	0	30	0	143	0	801	0
Total	34	14	475	23	6.026	38	98.209	58

Explicit solutions for the ES-Problem: An Overview

	Discrete Model	Continuous Model
Complete Markets	**Closed-form solution** Schulmerich/Trautmann (2001) Schulmerich (2001)	**Closed-form solution** Föllmer/Leukert (2000)
Incomplete Markets	**Numerical Solution and Approximation resp.** Schulmerich/Trautmann (2001) Schulmerich (2001)	**Numerical Approximation** Schulmerich (2001)

6. Conclusions

- ES-hedging is a reasonable alternative to classical approaches (superhedging, mean-variance-hedging) for hedging contingent claims in *incomplete* markets.
- Closed-form solutions are only for complete markets available.
- Calculating ES-strategies in discrete models is equivalent to the iterative solution of linear programs whose number increases *exponentially* with respect to the number of trading dates.
- LES-strategies approximate ES-strategies quite accurately.
- Calculating LES-strategies in discrete models is equivalent to the iterative solution of linear programs whose number increases only *linearly* with respect to the number of trading dates.