Sequential and Unrestricted Exercise of Warrants and Convertible Bonds: For whom does it pay?

by

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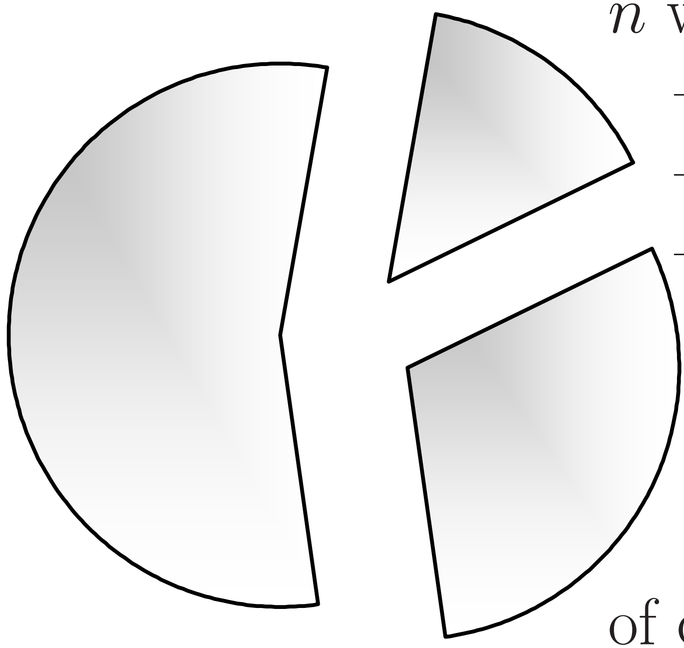
Outline:

1. Block exercise, unrestricted exercise and sequential exercise
2. Unrestricted exercise of European-type warrants
3. Sequential exercise of American-type warrants
4. Price impact of the block exercise constraint
5. Conclusion

firm value V_t

arbitrary stochastic process

debt $D_t(V_t)$
– face value F
– zero coupon bond
– maturity T_D



n warrants with
– total value $nW_t(V_t)$
– maturity $T < T_D$
– strike price K

N shares
of common stock with
total value $NS_t(V_t)$

$$\Rightarrow V_t = NS_t(V_t) + nW_t(V_t) + D_t(V_t)$$

Assumptions:

- Warrant exercise only at $t = T$ (European-type warrant) and exercise at $t = 0$ or $t = T$ (American-type warrant), respectively:

  one new share per warrant (for strike price K)

- Reinvestment of exercise proceeds in the same risk class

- No taxes, no transaction costs, no arbitrage

- No regular dividend payments

- $m \in [0, n]$ exercise policy of the warrantholders at time $t = T$

- V_{T^-} last firm value before T

$$\Rightarrow \quad V_T = V_{T^-} + mK$$

- for all $t \in [T, T_D]$

$$V_t = \bar{S}_t(V_t) + D_t(V_t) = (N + m)S_t(V_t) + D_t(V_t)$$

and

$$V_{T_D} = \bar{S}_{T_D}(V_{T_D}) + \min\{F; V_{T_D}\}$$

- Common stock can be seen as a call option on the firm's assets, since

$$\bar{S}_{T_D}(V_{T_D}) = \max\{V_{T_D} - F; 0\}$$

- Therefore: $\Delta_T(V) = \frac{\partial}{\partial V}\bar{S}_T(V) \in (0, 1)$ ("Delta") and $\Gamma_T(V) = \frac{\partial^2}{\partial V^2}\bar{S}_T(V) \geq 0$ ("Gamma")

1. Block exercise, unrestricted exercise and sequential exercise

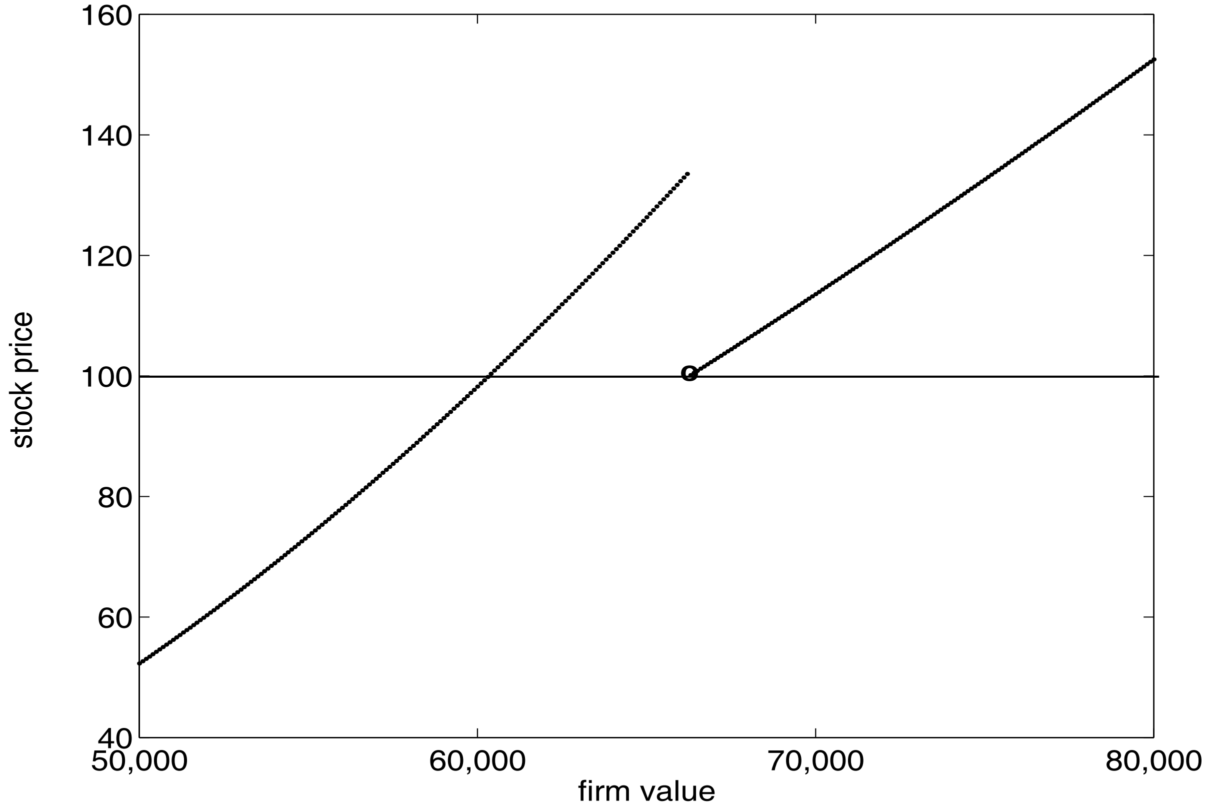
- **block exercise strategy** at maturity:

$$m = \begin{cases} 0 & \text{for} \quad \frac{1}{N+n}\bar{S}_T(V_{T^-} + nK) \; < \; K \\ n & \text{for} \quad \frac{1}{N+n}\bar{S}_T(V_{T^-} + nK) \; \geq \; K \, . \end{cases}$$

- **unrestricted exercise strategy**:
  all other strategies at maturity for European-type warrants

- **sequential exercise strategy**:
  some of the American-type warrants are exercised before maturity

Stock price by Block exercise strategy



Assumption: V_t follows a geometric Brownian motion
Parameters:
$r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$.

Key results of related literature

- **Ingersoll (1977):**
  "Sequential exercise can be optimal for a monopolistic warrantholder."
  (additional debt is not considered)

- **Spatt and Sterbenz (1988):**
  - *"There are reinvestment policies for which sequential exercise is not advantageous."*
  - *"Sequential exercise may be advantageous for monopoly and oligopoly warrantholders."*

  (additional debt is not considered and the magnitude of the advantage not analysed)

- **Bühler and Koziol (2003):**
  "Unrestricted exercise can be optimal for pricetakers in the presence of additional debt."
  (market structures with non-pricetakers are not analysed)

2. Unrestricted exercise of European-type warrants

Noncooperative game

- I set of warrantholders and P measure on I
- Warrantholder $i \in I$ holds n_i warrants with $n = \int_I n_i dP$
- Warrantholder $i \in I$ exercises $m_i \in [0, n_i]$ warrants with $m = \int_I m_i dP$ at time $t = T$ (m_{-i} exercise policy of all warrantholders without i)
- Payoff function of warrantholder $i \in I$ with $P(\{i\}) = 0$ (warrantholder i is non-atomic player/ pricetaker)

$$\pi_i(m_i, m_{-i}, V_{T^-}) = \frac{m_i}{N + m} \bar{S}_T(V_{T^-} + mK) - m_i K \,.$$

- Payoff function of warrantholder $A \in I$ with $P(\{A\}) = 1$ (warrantholder A is atomic player/ non-pricetaker)

$$\pi_A(m_A, m_{-A}, V_{T^-}) = \frac{m_A}{N + m_A + m_{-A}} \bar{S}_T(V_{T^-} + m_A K + m_{-A} K) - m_A K \,.$$

Nash equilibrium

$(m_i^*)_{i \in I}$ is a Nash equilibrium, if

$$\pi_i(m_i^*, m_{-i}^*, V_{T^-}) \geq \pi_i(m_i, m_{-i}^*, V_{T^-})$$

for all $i \in I$ and $m_i \in [0, n_i]$.

Non-atomic game:

- all warrantholders are pricetakers and $\int_I 1 dP = 1$
- all warrantholders have the same number of warrants, i.e. $n_i = n_j$ for all $i, j \in I$

Exercise policies of pricetakers

The strategy

$$(m_i^*, m_{-i}^*) = \begin{cases} (0,0) & \text{for} \quad V_{T^-} \in [0, \underline{V}) \\ (x^*, x^*) & \text{for} \quad V_{T^-} \in [\underline{V}, \overline{V}) \\ (n_i, n) & \text{for} \quad V_{T^-} \in [\overline{V}, \infty) \end{cases}$$

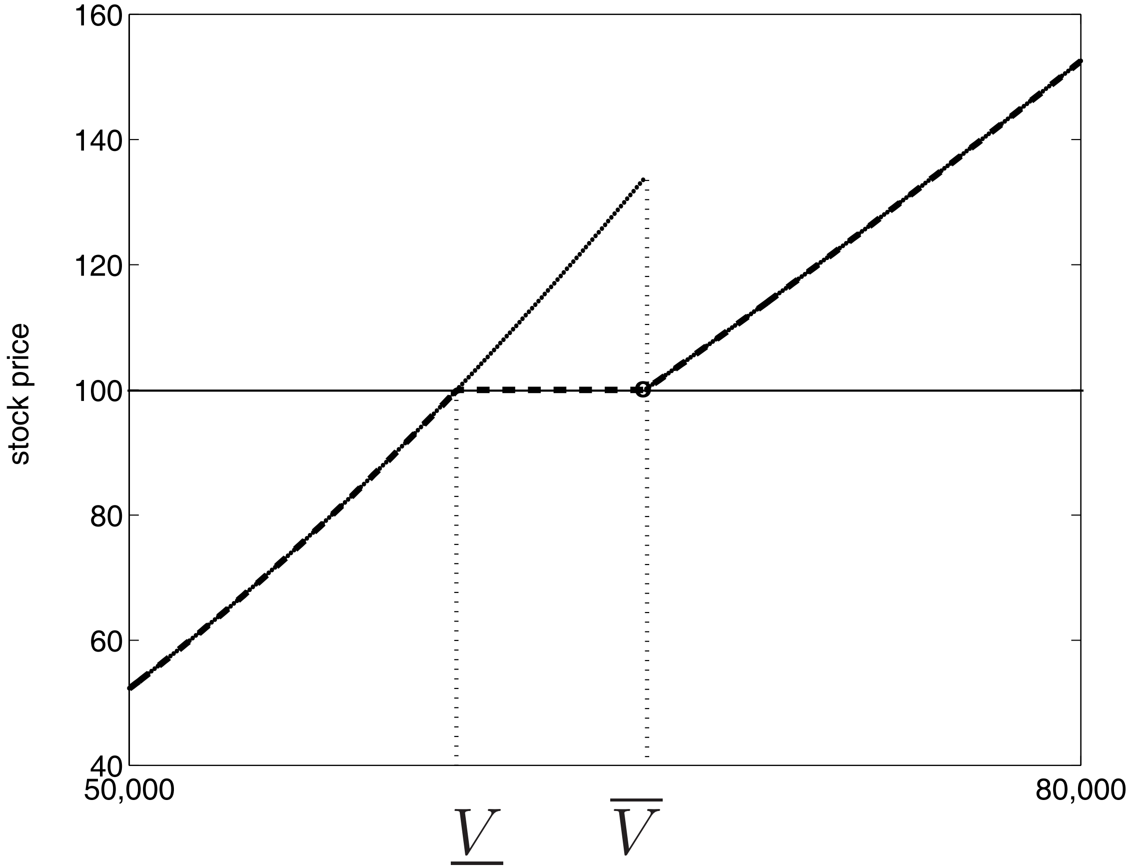
is a Nash equilibrium with

$$\frac{1}{N}\bar{S}_T(\underline{V}) = K \qquad \text{and} \qquad \frac{1}{N+n}\bar{S}_T(\overline{V} + nK) = K$$

and

$$\frac{1}{N+x^*}\bar{S}_T(V_{T^-} + x^* K) = K \, .$$

Stock price in the non-atomic game



Assumption: V_t follows a geometric Brownian motion
Parameters:
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Exercise policies of non-pricetakers

One-atomic game:

$A \in I$ warrantholder with $P(\{A\}) = 1$ and $n_A > 0$
All other warrantholders are pricetakers with $\int_{I \setminus \{A\}} 1 dP = 1$ and $n_i = n_j = n_{-A}$

The strategy

$$(m_A^*, m_{-A}^*) = \begin{cases} (0,0) & \text{for } V_{T^-} \in [0, \underline{V}) \\ (0, x_{-A}^*) & \text{for } V_{T^-} \in [\underline{V}, \underline{V}_A) \\ (x_A^*, n_{-A}) & \text{for } V_{T^-} \in [\underline{V}_A, \overline{V}_A) \\ (n_A, n_{-A}) & \text{for } V_{T^-} \in [\overline{V}_A, \infty) \end{cases}$$

is a Nash equilibrium with

$$K = \frac{1}{N + x_{-A}^*} \bar{S}_T(V_{T^-} + x_{-A}^* K)$$

$$K = \frac{N + n_{-A}}{(N + n_{-A} + x_A^*)^2} \bar{S}_T(V_{T^-} + n_{-A}K + x_A^* K) + \frac{x_A^*}{N + n_{-A} + x_A^*} K \Delta_T(V_{T^-} + n_{-A}K + x_A^* K) \,.$$

Two-atomic game:

Two warrantholders with $n_B + n_b = n$ and $n_B \geq n_b$

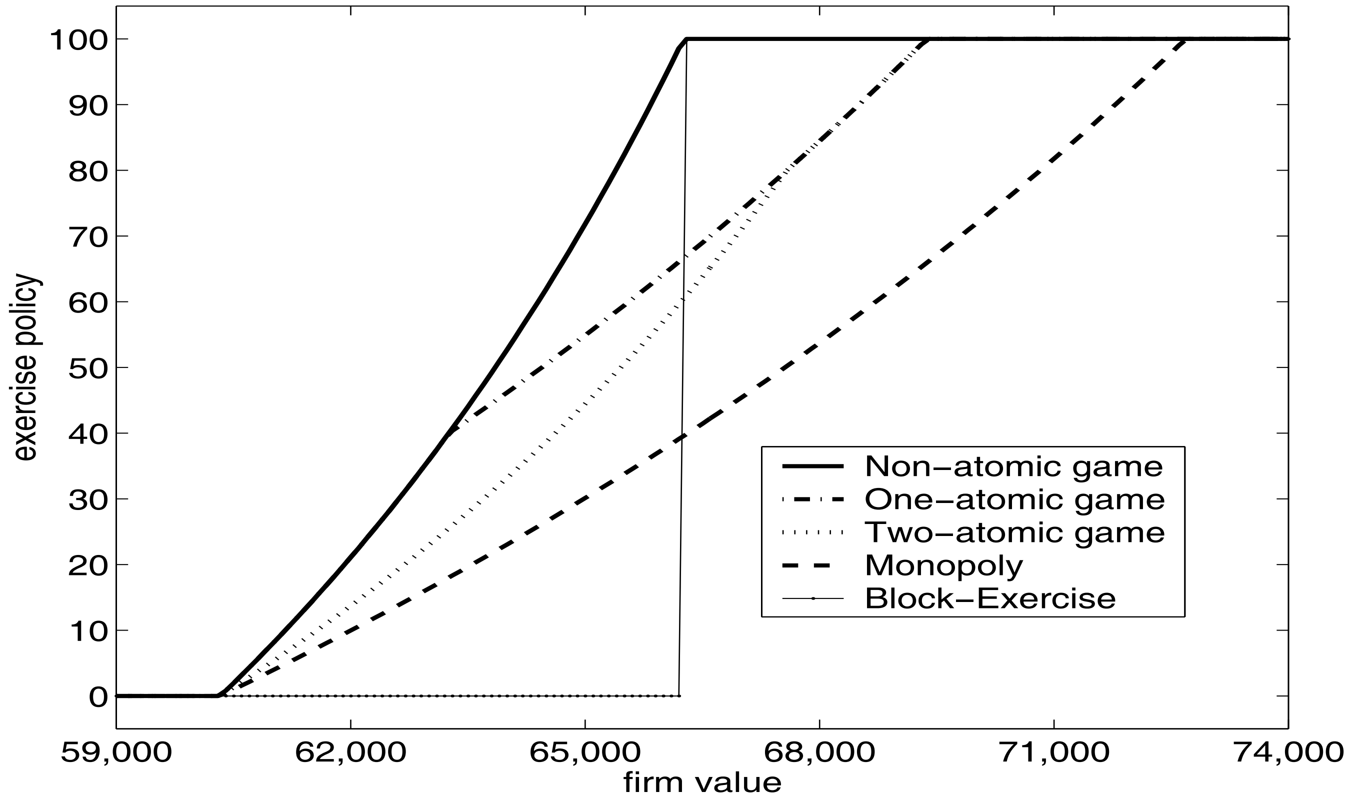
The strategy

$$(m_b^*, m_B^*) = \begin{cases} (0,0) & \text{for} \quad V_{T^-} \in [0, \underline{V}) \\ (x_b^*, x_b^*) & \text{for} \quad V_{T^-} \in [\underline{V}, \overline{V}_b) \\ (n_b, x_B^*) & \text{for} \quad V_{T^-} \in [\overline{V}_b, \overline{V}_B) \\ (n_b, n_B) & \text{for} \quad V_{T^-} \in [\overline{V}_B, \infty) \end{cases}$$

is a Nash equilibrium with

$$\begin{aligned} K &= \frac{N + x_b^*}{(N + 2x_b^*)^2} \bar{S}_T(V_{T^-} + 2x_b^* K) + \frac{x_b^*}{N + 2x_b^*} K \Delta_T(V_{T^-} + 2x_b^* K) \\ K &= \frac{N + n_{-b}}{(N + n_{-b} + x_B^*)^2} \bar{S}_T(V_{T^-} + n_{-b}K + x_B^* K) + \frac{x_B^*}{N + n_{-b} + x_B^*} K \Delta_T(V_{T^-} + n_{-b}K + x_B^* K) \,. \end{aligned}$$

Optimal exercise policy



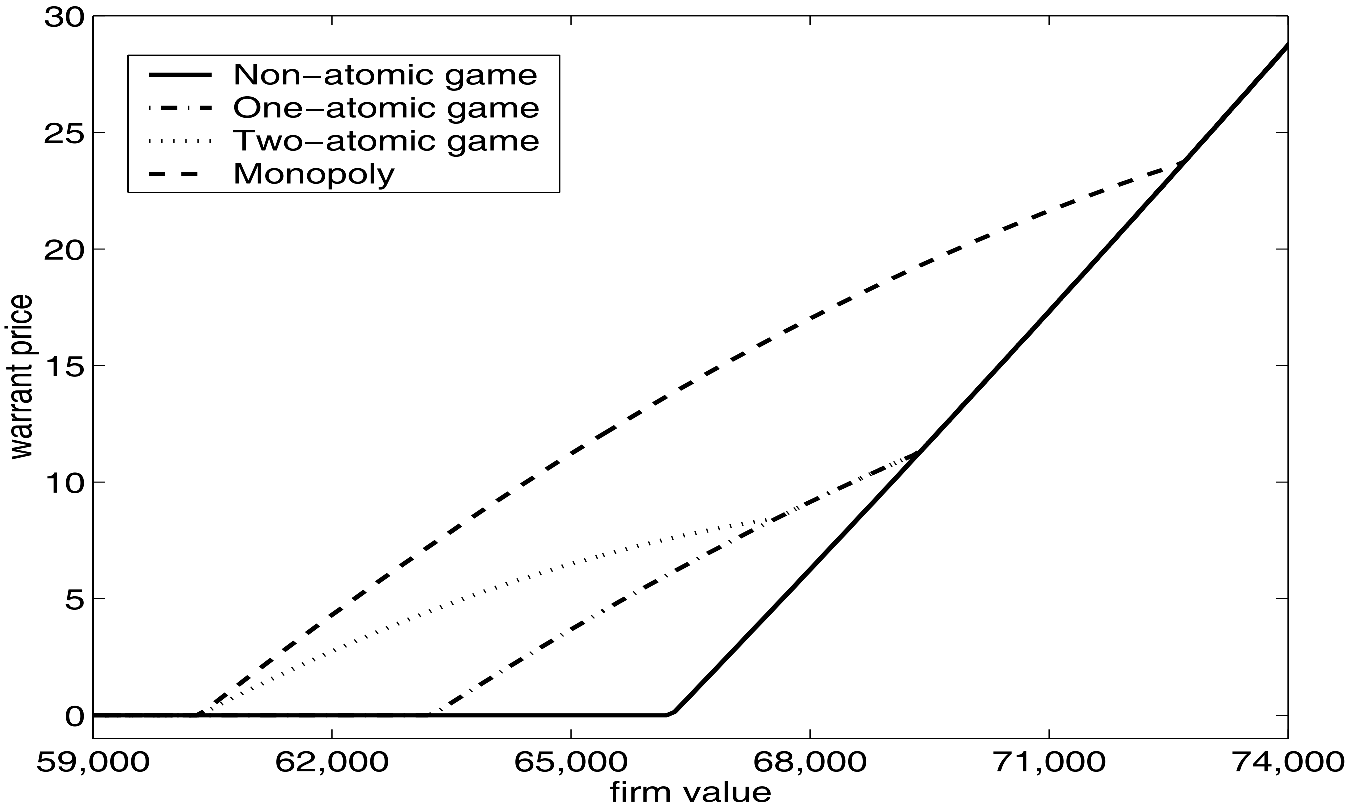
Assumption: V_t follows a geometric Brownian motion

Parameters:

$r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$.

$n_A = n_B = 60$ and $n_{-A} = n_b = 40$

Exercise values of European-type warrants



Assumption: V_t follows a geometric Brownian motion
Parameters:
$r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$.
$n_A = n_B = 60$ and $n_{-A} = n_b = 40$

3. Sequential exercise of American-type warrants

Rescaling the firm's investment

- Exercise in $t = 0$ or $t = T$
- $m \in [0, n]$ exercise policy in $t = 0$; sales of $n - m$ warrants to pricetakers
- $m_T \in [0, n - m]$ exercise policy in $t = T$ of pricetakers
- All warrantholders know V_0 and probability measure Q of random variable V_T
- Investment of exercise proceeds in the same risk class ("rescaling")
- Current stock price is given by

$$S_0(V_0, m) = e^{-rT} \int_{\mathbb{R}_+} S_T \left(\frac{V_0 + mK}{V_0} V_T + m_T(V_T) K \right) dQ$$

Example: Sequential exercise

Assumption: V_t follows a geometric Brownian motion with $V_0 = 65,000$
Parameters: $r = 0\%$, $\sigma = 0.3$, $F = 15,000$, $T_D = 5.5$, $T = 0.75$, $N = 50$, $n = 50$ and $K = 250$.

- *non-atomic game:* 50 warrantholders with each 1 warrant exercise no warrants
- *one-atomic game:* 25 warrantholders with each 1 warrant exercise no warrants
  1 warrantholder with 25 warrants exercises 23 warrants
- *two-atomic game:* 2 warrantholders with each 25 warrants exercise each 17 warrants
- *monopoly:* 1 warrantholder with 50 warrants exercises 50 warrants

	Non-atomic game	One-atomic game	Two-atomic game	Monopoly
stock price	625.63	625.68	625.70	625.76
warrant price	375.64	375.74	375.81	375.96
debt value	14,936.41	14,932.54	14,930.05	14,924.49

Bounds on the warrant's price sensitivity

- Warrant price $W_0(V_0, m)$ is an increasing and convex function of the number of warrants exercised, m.

- *Lower bound:* of partial derivative

$$\frac{1}{n_A} K \left(1 - e^{-rT}\right) \;\leq\; \frac{\partial}{\partial m} W_0(V_0, m^*)$$

- *Upper bound:* of partial derivative

$$\frac{\partial}{\partial m} W_0(V_0, m^*) \;\leq\; \frac{1}{n_A} K \left(1 - e^{-rT}\right) + \frac{1}{n_A} K Q(\{V_T \leq \underline{V}_T(m^*)\})\,.$$

  $\underline{V}_T(m^*)$: highest firm value in T without any warrant exercise

- No warrant is exercised, if the interest rate r is sufficiently high, as the marginal payoff is bounded by

$$\frac{\partial}{\partial m_A} \pi_A(m_A, m) \;\leq\; \frac{n_A}{N+n} K \frac{W_0^{am}(V_0)}{V_0} - K \left(1 - e^{-rT}\right)\,.$$

  where W_0^{am} is the price of an at-the-money warrant with maturity T

Example: Monopolistic warrantholder

- Assumption: V_t follows again a geometric Brownian motion
  Parameters: $r = 1\%$, $\sigma = 0.4$, $V_0 = 63,000$, $F = 15,000$, $T_D = 7$, $T = 1$, $N = 50$, $n = 50$ and $K = 250$.

- Exercise policy of a monopolistic warrantholder:
  $m^* = 4$ with $W_0(V_0, 4) - W_0(V_0, 3) = 373.64 - 373.57 = 0.07$.

- Bounds on the warrant price sensitivity:
  – Lower bound:
  $$\frac{1}{n_A} K \left(1 - e^{-rT}\right) = 0.0498$$
  – Upper bound:
  $$\frac{1}{n_A} K \left(1 - e^{-rT}\right) + \frac{1}{n_A} K Q(\{V_T \leq \underline{V}_T(m^*)\}) = 0.0629 + 0.0498 = 0.1127\,.$$

- Absolute difference by exercise of 4 warrants:
  less than $0.4508 = 0.12\%$ of $W_0(V_0, m^*)$.

Investment in zero bonds

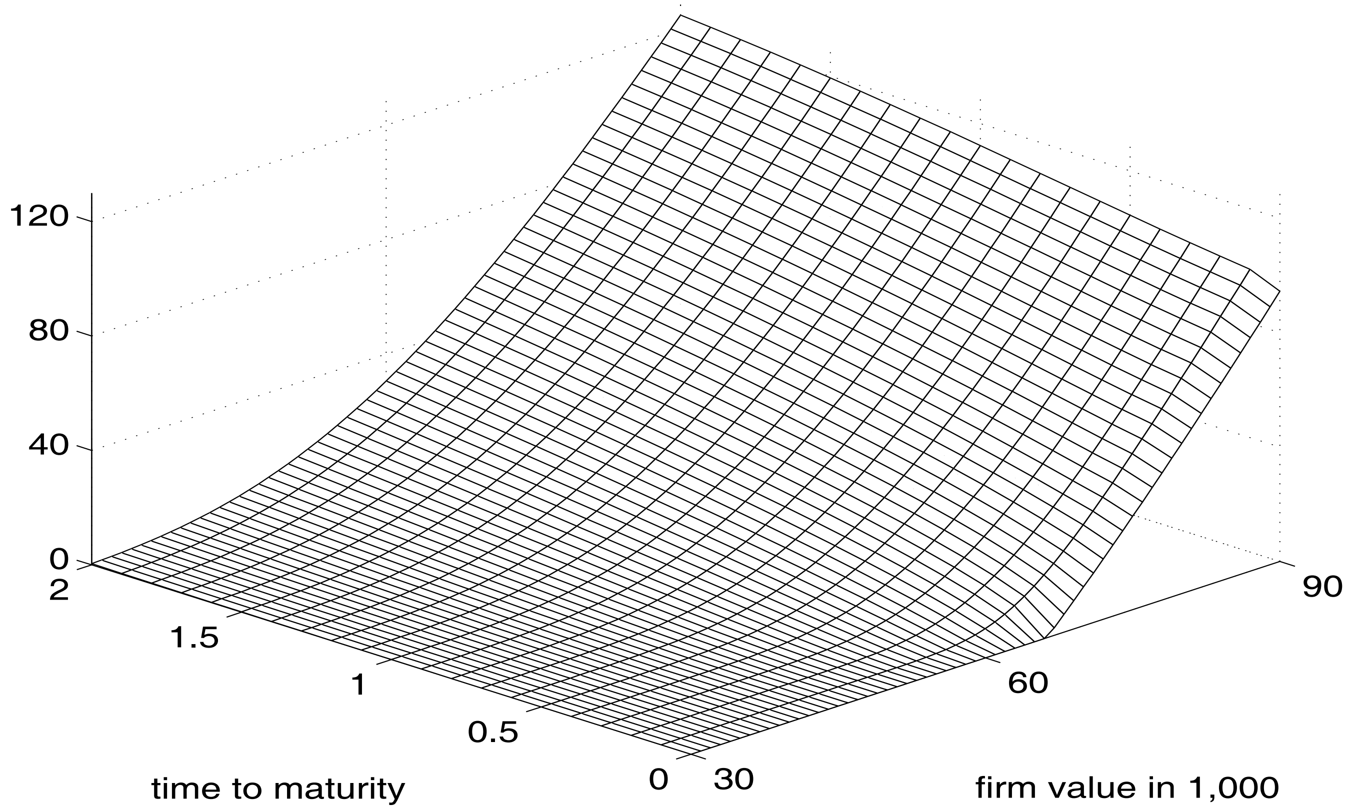
- Investment of exercise proceeds in zero coupon bonds
- Stock price is given by

$$S_0(V_0, m) = e^{-rT} \int_{\mathbb{R}_+} S_T \left(V_T + e^{rT} mK + m_T(V_T) K \right) dQ$$

- Payoff function $\pi_i(\cdot)$ is decreasing w.r.t. m_i for all $i \in I$
- Optimal exercise policy is $m_i^* = 0$ for all $i \in I$

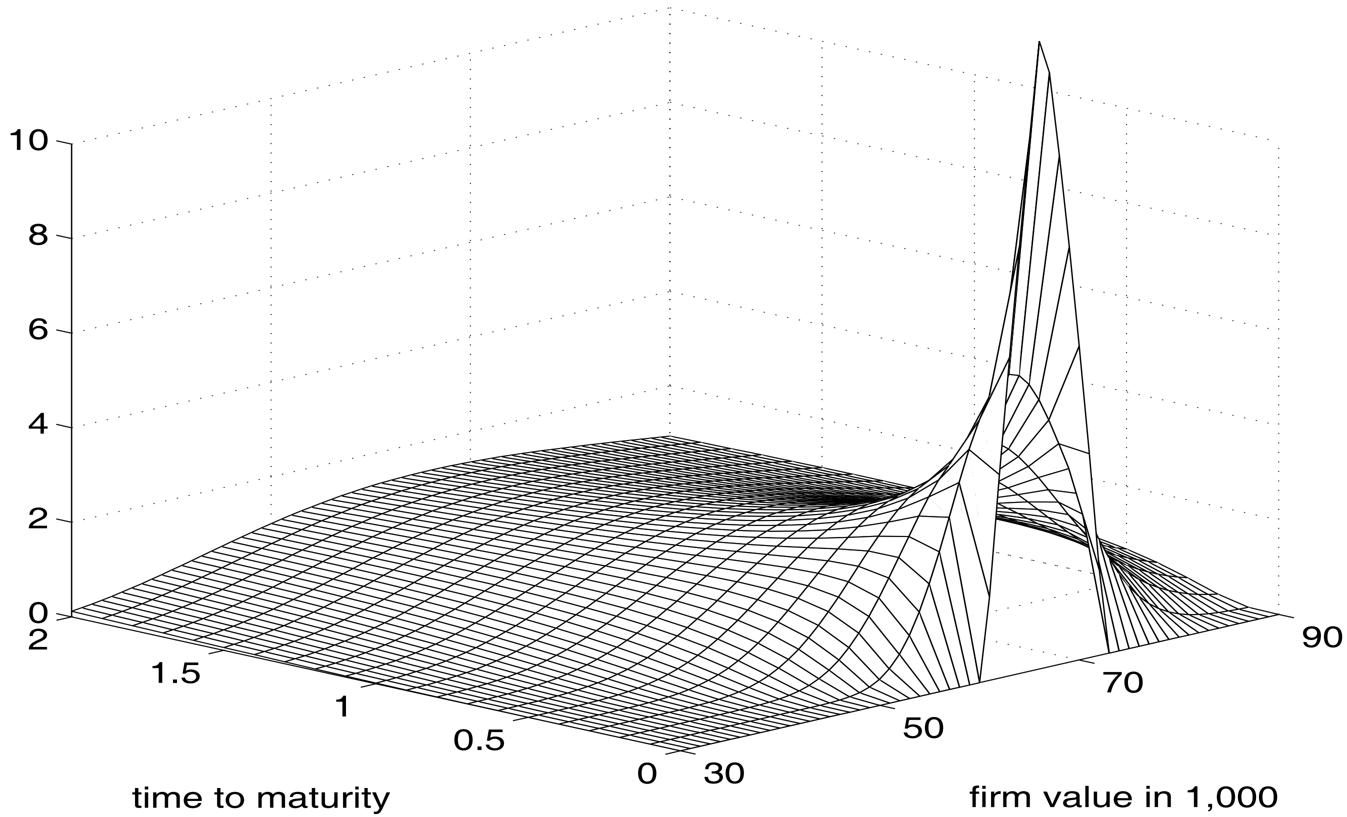
4. Price impact of the block exercise constraint

- Warrant price in the presence of pricetakers and block exercise



Assumption: V_t follows a geometric Brownian motion
Parameters:
$r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$.

- Absolute price differences:
  warrant price (monopoly) - warrant price (pricetakers)



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Parameters:
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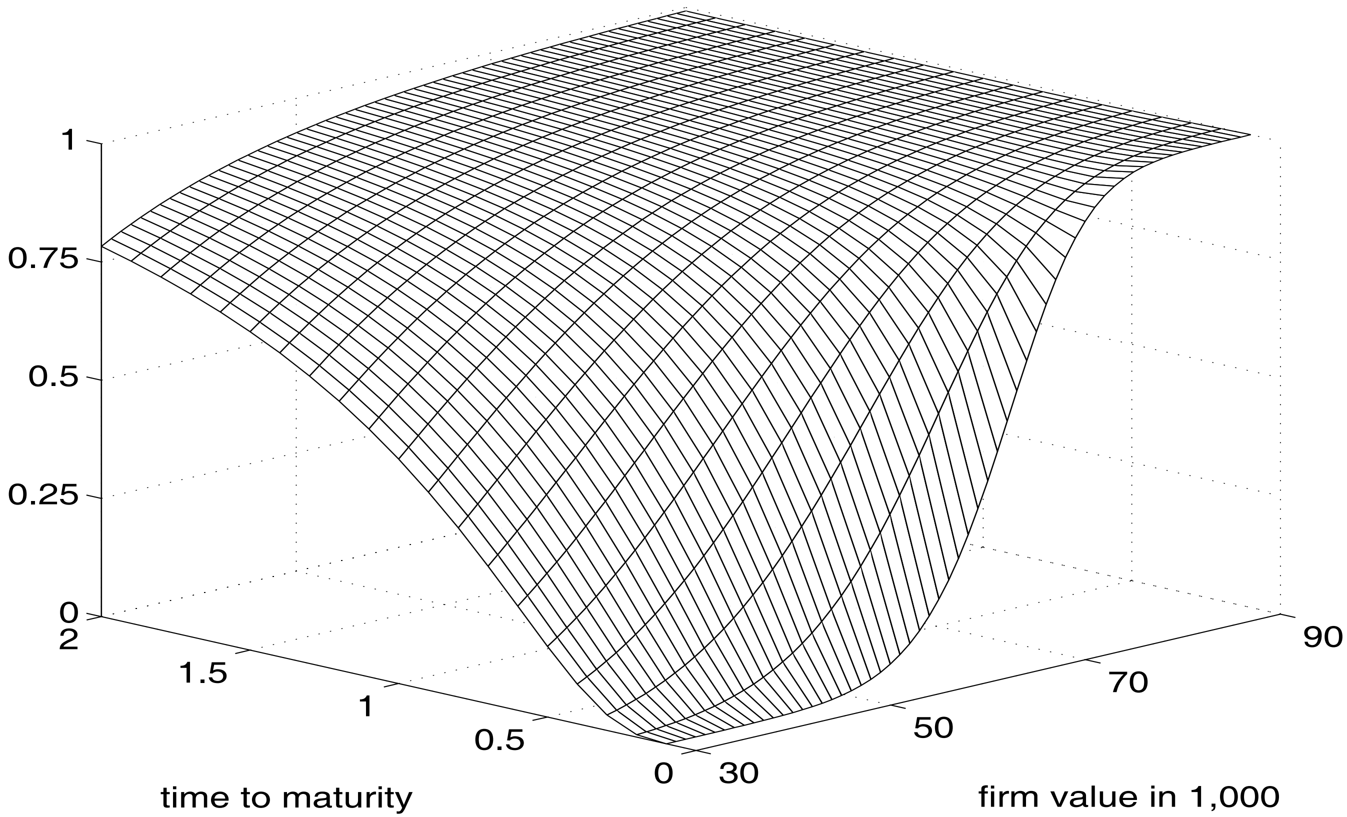
$\dfrac{\textbf{warrant price (pricetakers)}}{\textbf{warrant price (monopoly)}}$



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Parameters:
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Volatility of Equity

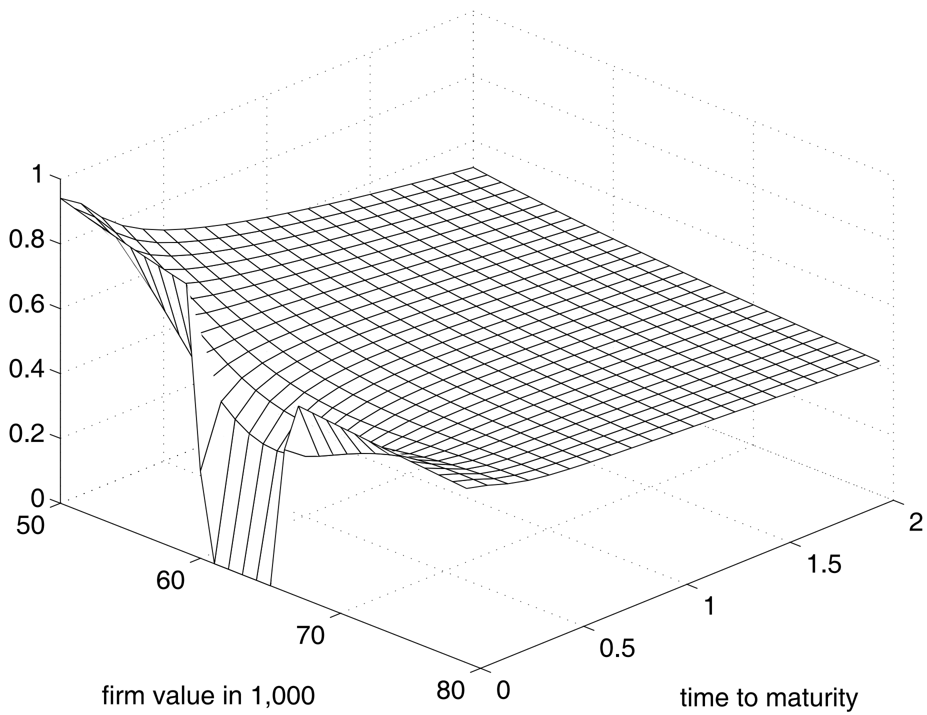
σ_V volatility of the firm value, σ_S volatility of the equity

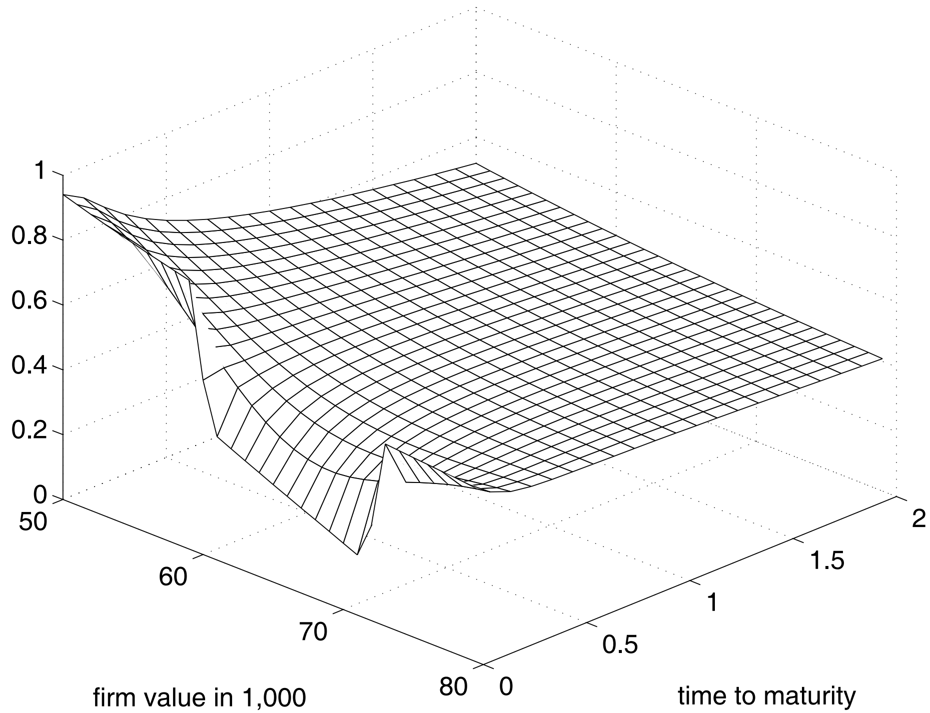
$m^*(V_T)$ optimal exercise rate of the warrantholders in time T

$$\begin{aligned}\sigma_S &= \sigma_V \frac{\partial S_t(V_t)}{\partial V_t} \frac{V_t}{S_t(V_t)} \\ &= \sigma_V \left(\int_0^\infty \Delta_T(V_{T^-} + m^*(V_{T^-})K) dQ \right) \left(\frac{V_t}{\int_0^\infty \frac{\bar{S}_T(V_{T-} + m^*(V_{T-}))}{N + m^*(V_{T-})} dQ} \right)\end{aligned}$$

Volatility of Equity

pricetakers versus monopoly





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Parameters:
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5. Conclusion

- Sequential exercise is either not optimal or leads to an insignificant price impact, if the exercise proceeds rescale the firm's investment.
- Unrestricted exercise (as opposed to block exercise) is beneficial in the presence of non-pricetaking warrantholders.
- In the presence of monopoly or oligopoly warrantholders, warrants are traded between pricetakers for a higher price.
- Unrestricted exercise pays for all warrantholders in the presence of non-pricetakers.