Local Expected Shortfall-Hedging in Discrete Time

by

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2. Model Framework
3. Hedging approaches
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1. Introduction

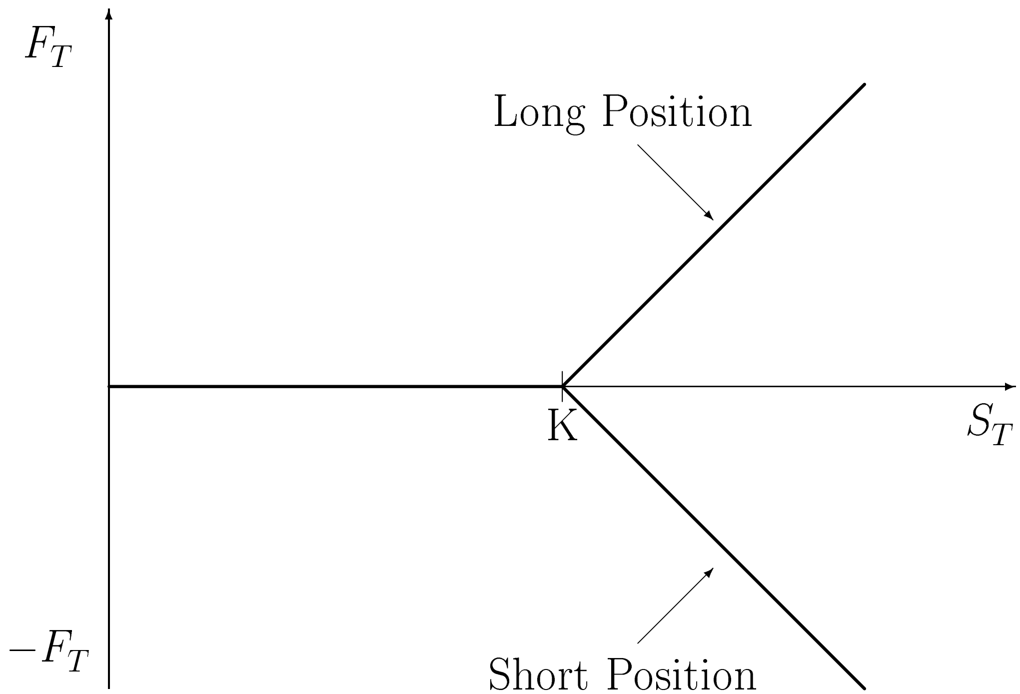
- Let us assume a situation where an investor has written a European call option on a stock, say for a price of 10 €.

- Suppose that the market is incomplete and that the investor is unwilling to follow a superhedging strategy which requires very often to buy one unit of the underlying instrument for, say 100 €.

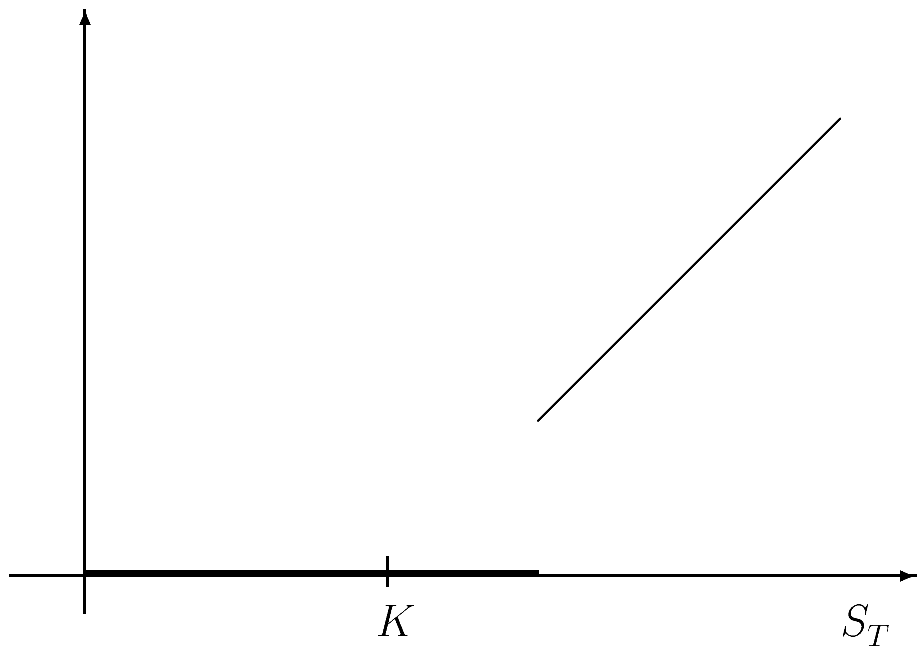
- **Question: What is the optimal self-financing hedging strategy under a constraint on the initial hedging capital?**

- Föllmer/Leukert (2000) propose a self-financing hedging strategy which minimizes the **expected shortfall** in a Black/Scholes (1973) model. This approach is in the spirit of the martingale approach of portfolio optimization.

Hedging Object: Short position of a European call with exercise price K, expiration date T, terminal value $F_T = (S_T - K)^+$, and present value F_0.



- **Föllmer/Leukert (2000):** Minimizing the investor's expected shortfall under a budget constraint is in the Black/Scholes model tantamount to (super-) hedge a suitable gap option ("modified claim").



- **Question:** Why should an investor not follow a replication strategy if the market is frictionless and complete? (Risk-averse investors would generally follow a perfect hedging strategy in a complete markets setting like the one assumed by Föllmer/Leukert (2000)).

2. Model Framework

We assume a situation where an investor has written a European contingent claim on a stock and wants to hedge the occuring risk with a fixed but arbitrary initial hedging capital $\bar{V}_0$.

- **Hedging Object:** Short position of a European contingent claim F_T.
- **Hedging Instruments:**
  - Underlying stock $S = (S_0, S_1, \ldots, S_T)$ and
  - riskless money market account $B_t = (1+r)^t$, $t = 0, 1, \ldots, T$.
- **Hedging Strategies:** To hedge the contingent claim the investor chooses a strategy $H = (h, h^0)$ where $h_t (h_t^0)$ represents the quantity of the stock (money market account) held in the portfolio at time t.

  The value of a hedging strategy is $V_t(H) = h_t \cdot S_t + h_t^0 \cdot B_t$.

  The set of all self-financing strategies is denoted by $\mathcal{H}_{\mathbf{S}}$.

3. Hedging Approaches

Risk measures:

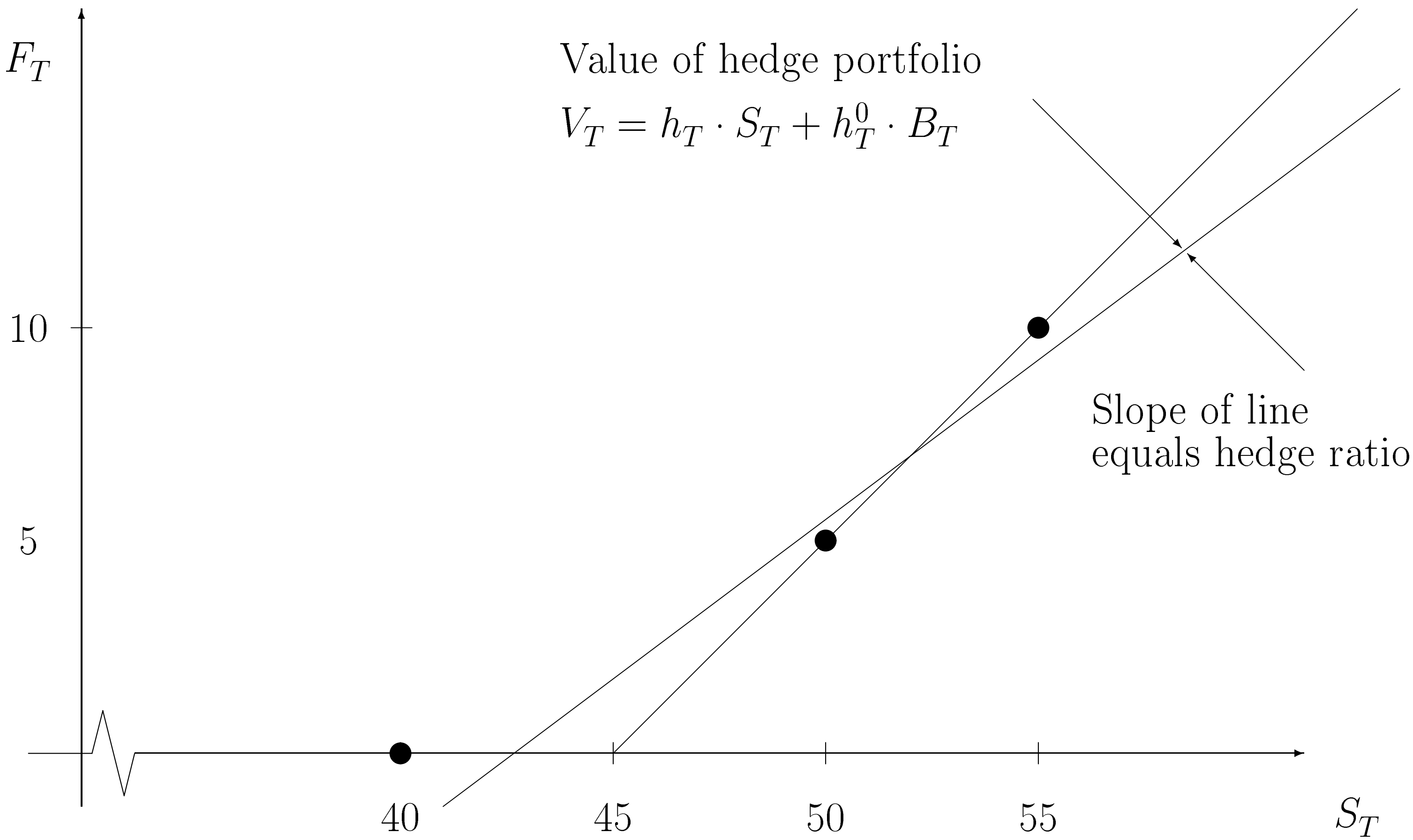
- Two-sided risk measures
  - Variance
  - Standard deviation
- One-sided risk measures (Shortfall risk measures)
  - Shortfall Probability, Value-at-Risk (not a coherent risk measure, $\rightarrow$ Quantile Hedging)
  - Expected Shortfall (an "almost" coherent risk measure)

Motivation for shortfall-based hedging approaches:

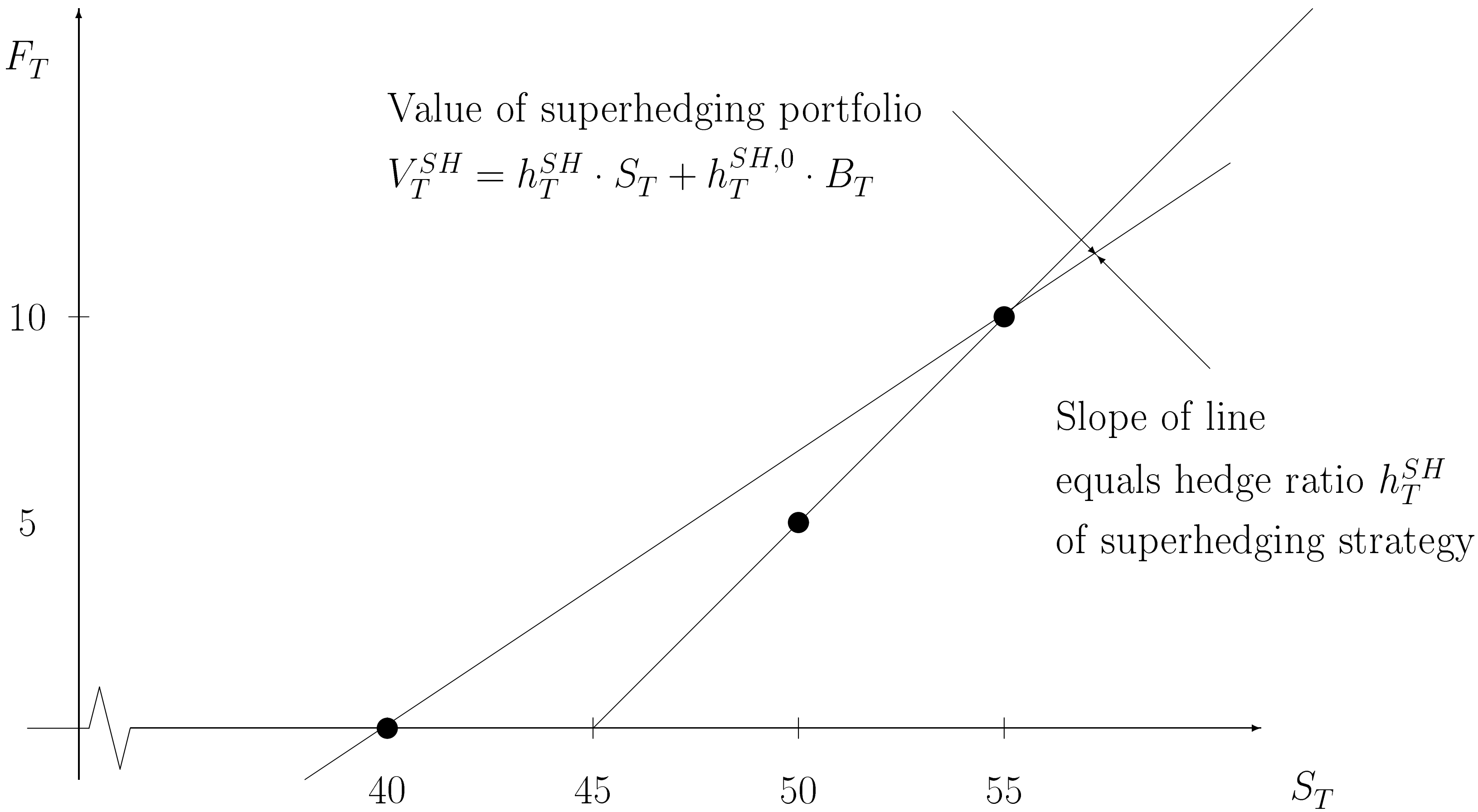
- In complete markets: hedger is not willing to invest completely the proceeds from writing the option.
- In incomplete markets: hedger is not willing or able to finance a superhedging strategy.

<table>
<tr><td></td><td>Complete Markets</td><td>Incomplete Markets</td><td></td></tr>
<tr><td>No Shortfall Risk</td><td>Delta-Hedging:
Black/Merton/Scholes (1973)
Cox/Ross/Rubinstein (1979)</td><td>Superhedging:
El Karoui/Quenez (1995)
Naik/Uppal (1992)</td><td rowspan="3">No Restriction on Initial Hedging Capital</td></tr>
<tr><td rowspan="3">Shortfall Risk</td><td></td><td>Local Risk-Hedging:
Föllmer/Schweizer (1991)
Schweizer (1992)</td></tr>
<tr><td colspan="2">Global Variance-Hedging: Schweizer (1996)</td></tr>
<tr><td colspan="2">Shortfall Probability-Hedging:
Föllmer/Leukert (1999)
Global Expected Shortfall-Hedging:
Föllmer/Leukert (2000), Cvitanić/Karatzas (1999)
Cvitanić (1998), Schulmerich/Trautmann (2001),
Schulmerich(2001)
Local Expected Shortfall-Hedging:
Schulmerich/Trautmann (2001), Schulmerich(2001)</td><td>Restriction on Initial Hedging Capital</td></tr>
</table>

Hedging in the trinomial model: A graphical illustration



Superhedging in the trinomial model: A graphical illustration



4. Expected-Shortfall Hedging

- **Problem ES:** Find a self-financing strategy H which minimizes

$$E_P[(F_T - V_T(H))^+] \text{ under the constraints } V_0(H) = \bar{V}_0 \text{ and } H \in \mathcal{H}_S.$$

- **Solution:** Föllmer/Leukert (2000), Cvitanić/Karatzas (1999), and Pham (1999) propose a two-step procedure similar to the martingale approach of portfolio optimization:

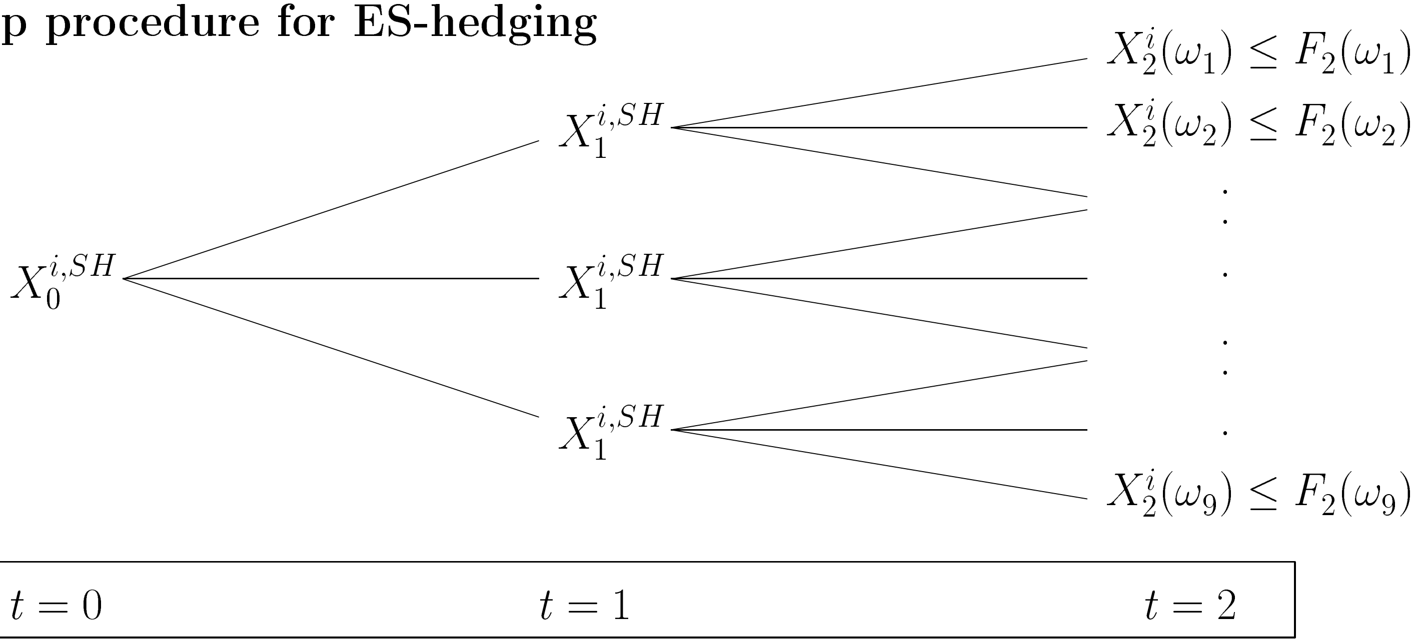
Step 1: **Static optimization problem:** (easy to solve if $\mathcal{Q}$ is a singleton)

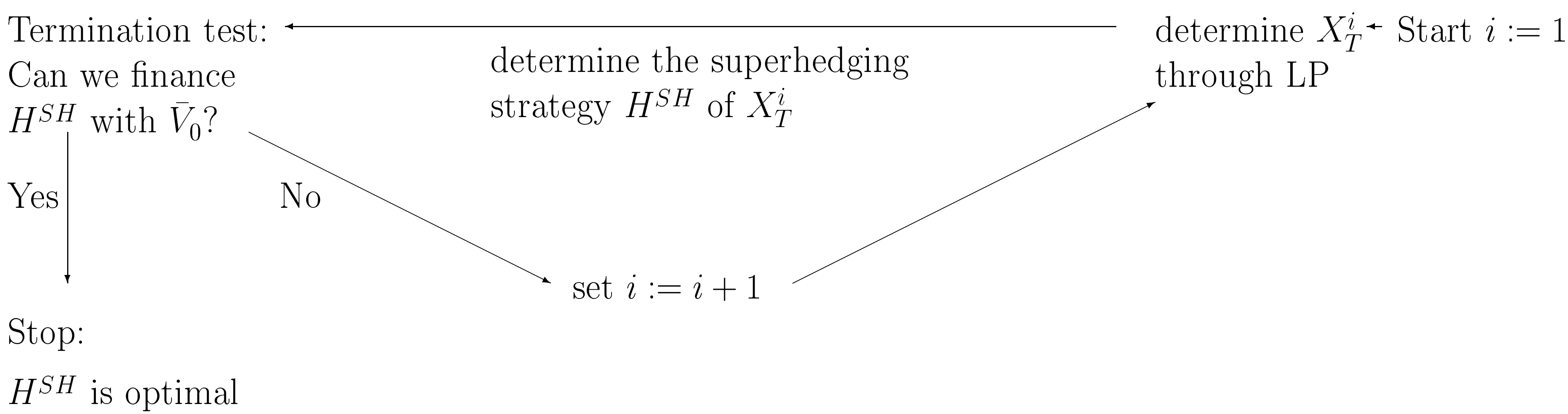
$$\max_X E_P(X) \text{ under the constraints } \sup_{Q \in \mathcal{Q}} E_Q(X/B_T) \leq \bar{V}_0 \text{ and } F_T \geq X.$$

Step 2: **Representation problem:**

Superhedge the modified claim X^* calculated in step 1.

The two-step procedure for ES-hedging





5. Local Expected Shortfall-Hedging

- **Idea:** We partition the complex overall problem ES into several one-period problems and minimize the expected shortfall only locally.

- **Problem LES:**

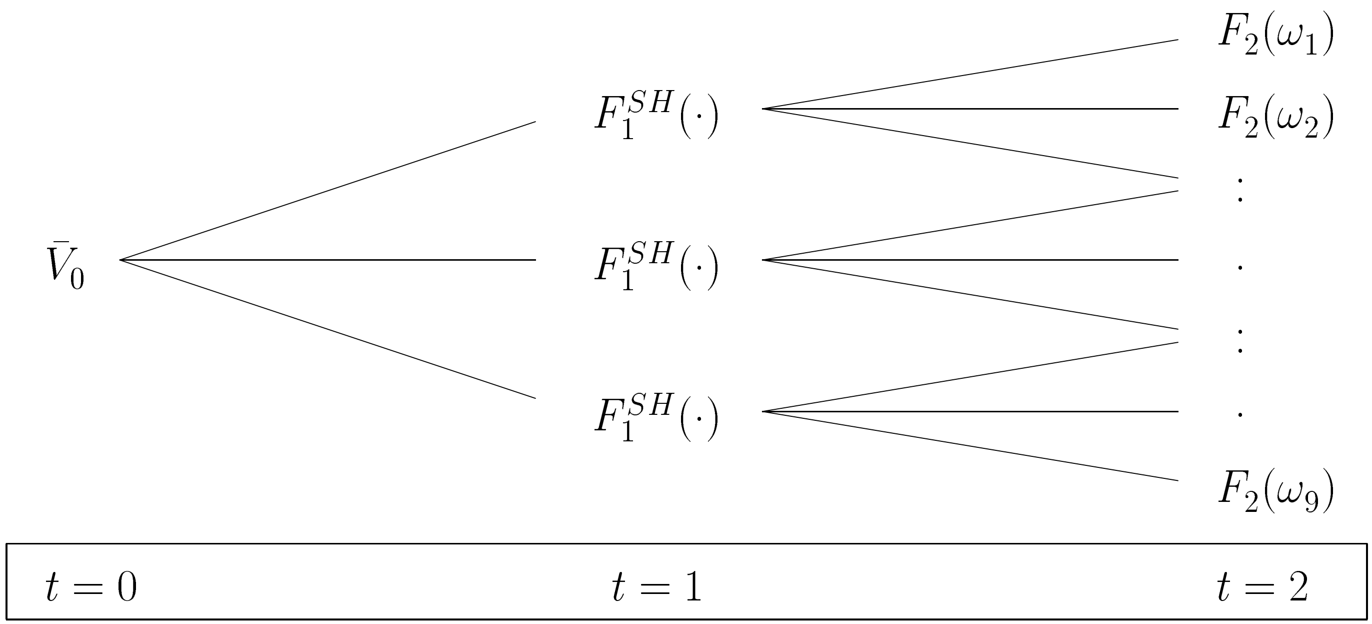
  Let $\mathcal{G}_t = \sigma(H_1^{LES}, \ldots, H_t^{LES})$ denote the σ-field generated by the LES-hedging strategy until time t. Then, find sequentially a self-financing strategy $H^{LES} = (H_1^{LES}, \ldots, H_T^{LES})$ with $V_0(H^{LES}) = \bar{V}_0$ whose components H_t^{LES} minimize the *(local) expected shortfall*

$$E_P[(F_t^{SH} - V_t(H))^+ \mid \mathcal{F}_{t-1} \vee \mathcal{G}_{t-1}] \quad \text{for} \quad t = 1, \ldots, T.$$

- **Solution:**

  Calculate ES-strategies in a one-period model n-times via the ES-algorithm!

The procedure for LES-hedging



Start $\longrightarrow$ $\longrightarrow$ Stop

determine H_0^{LES} through one-period ES-algorithm

determine H_1^{LES} through one-period ES-algorithm

Computational complexity of ES- and LES-strategies: Number of LP's to be solved

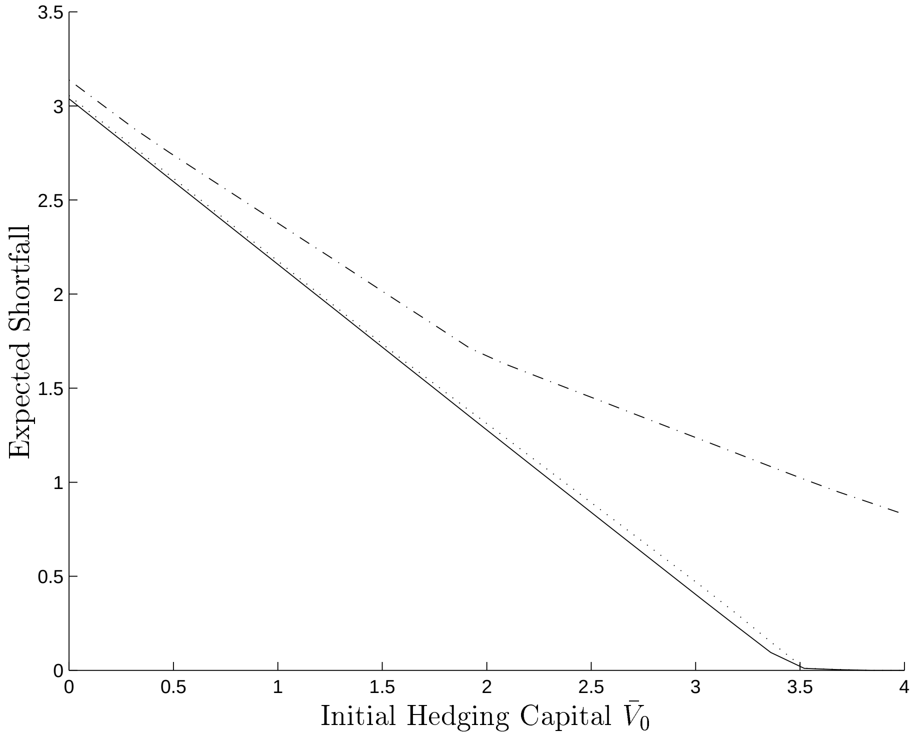
Number of constraints in linear programs	$n=2$ ES	$n=2$ LES	$n=3$ ES	$n=3$ LES	$n=4$ ES	$n=4$ LES	$n=5$ ES	$n=5$ LES
1	1	2	1	3	1	4	1	5
2	29	12	443	20	5.881	34	97.406	53
3	1	0	1	0	1	0	1	0
≥ 4	3	0	30	0	143	0	801	0
Total	34	14	475	23	6.026	38	98.209	58

Number of Periods: $n=2$ to $n=5$.

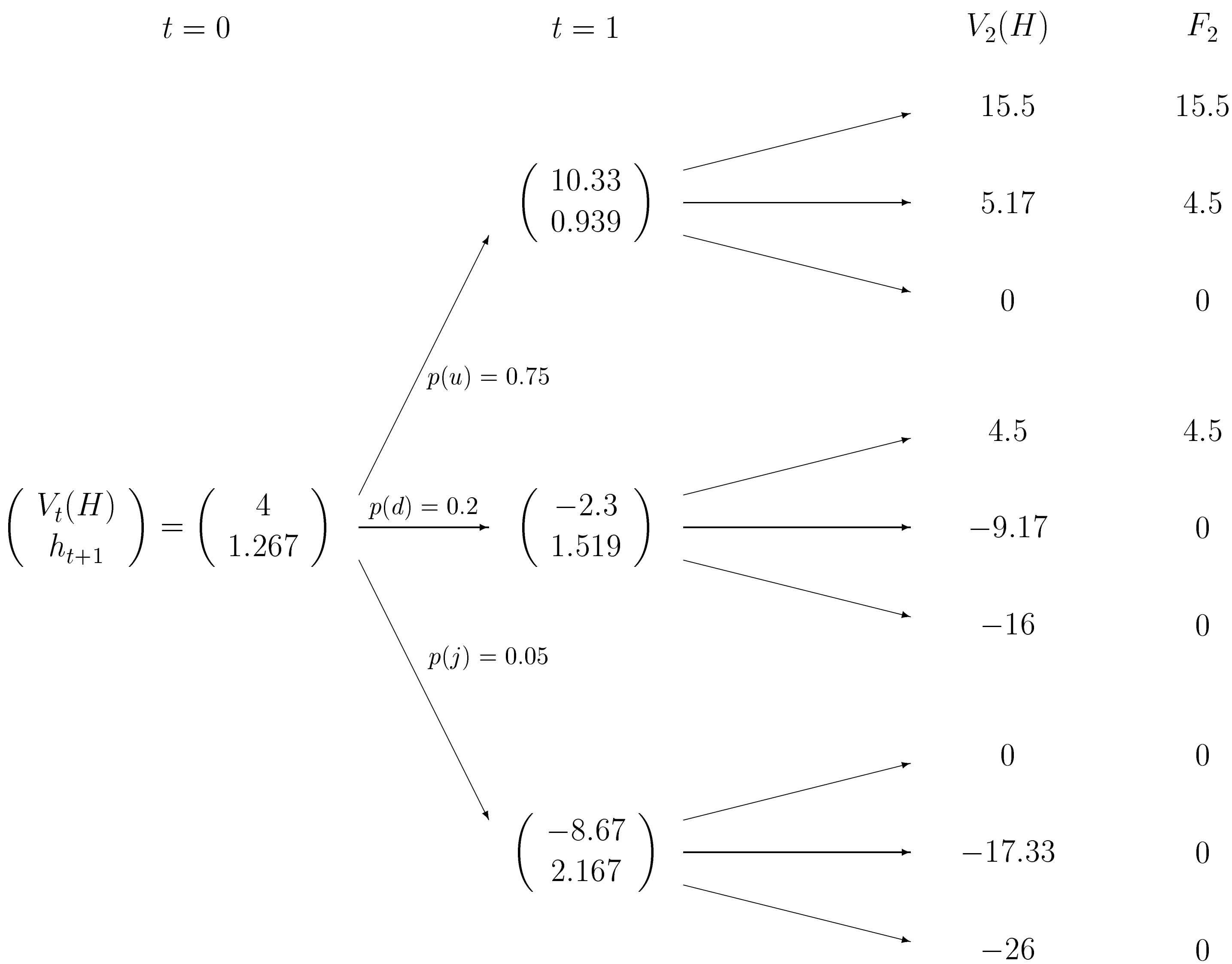
The efficient frontier of the ES- and LES-strategy

Expected Shortfall vs. Initial Hedging Capital

Parameter values: initial stock price = 50 €; annual interest rate (r) = 5 %; annual volatility of the “normal” stock price return (σ) = 20 %; annual expected rate of the “normal” return of the stock (α) = 15 %; time to maturity of the option (τ) = 1/12; strike price of the option (K) = 47 €; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 3.

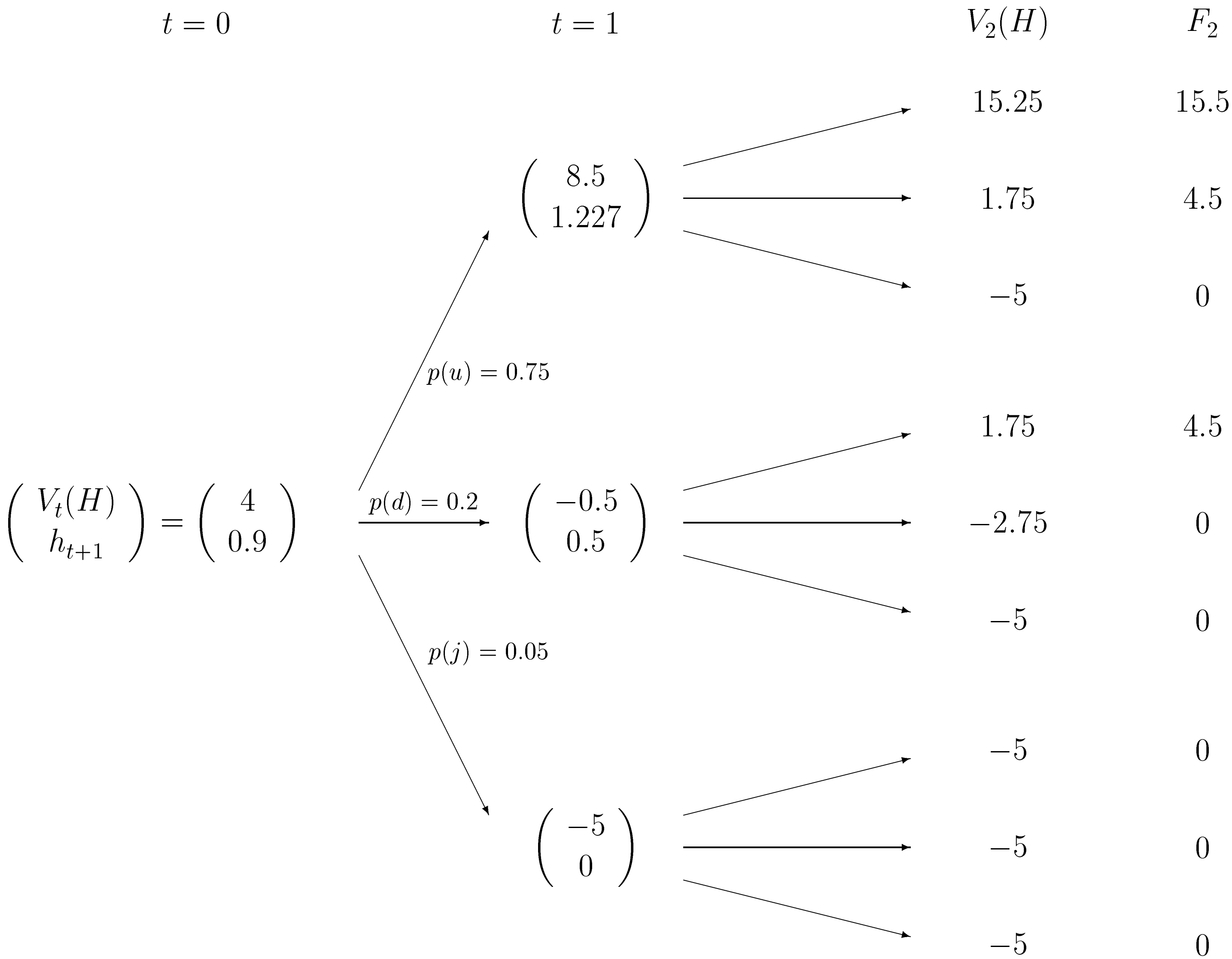


Example (LES-strategy without shortfall bound ($b = \infty$))



$E_P(F_2 - V_2(H))^+ = 0.765$

Example (LES-strategy with a shortfall bound ($b = 5$))



$$E_P(F_2 - V_2(H))^+ = 1.563$$

Distribution of the total hedging costs ($b_c = \infty$)

Parameter values: initial stock price = \$50; annual interest rate (r) = 5 %; annual volatility of the stock price return (σ) = 20 %; annual expected rate of return of the stock (α) = 15 %; time to maturity of the option (τ) = 0.25; strike price of the option (K) = \$47; expected number of jumps (λ)= 3 per year; number of trading periods (n) =10.

	Initial Hedging Capital					
	$\bar{V}_0 = 5$	$\bar{V}_0 = 4$	$\bar{V}_0 = 3$	$\bar{V}_0 = 2$	$\bar{V}_0 = 1$	$\bar{V}_0 = 0$
Mean	4.31	4.07	3.78	3.48	3.18	2.87
Std. Dev.	0.40	2.00	3.97	5.96	7.94	9.94
Minimum	3.43	2.43	1.43	0.43	-0.56	-1.56
5% Quantile	3.54	2.85	1.90	0.90	-0.10	-1.10
50% Quantile	4.34	3.61	2.72	1.73	0.88	-0.12
75% Quantile	4.60	3.97	2.97	1.99	1.14	0.31
90% Quantile	4.80	4.42	4.75	4.87	4.98	5.10
95% Quantile	4.85	5.84	8.44	10.90	13.35	15.82
99% Quantile	4.97	13.94	23.63	33.33	43.03	52.73
Maximum	5.00	106.95	208.58	310.22	411.86	513.50

Distribution of the total hedging costs ($\bar{V}_0 = 2$)

Parameter values: initial stock price = \$50; annual interest rate (r) = 5 %; annual volatility of the stock price return (σ) = 20 %; annual expected rate of return of the stock (α) = 15 %; time to maturity of the option (τ) = 0.25; strike price of the option (K) = \$47; expected number of jumps (λ)= 3 per year; number of trading periods (n) =10.

	Upper bound for the total hedging costs					
	$b_c = 6$	$b_c = 8$	$b_c = 10$	$b_c = 15$	$b_c = 20$	$b_c = 25$
Mean	4.17	4.08	4.01	3.91	3.85	3.81
Std. Dev.	2.02	2.99	3.64	4.63	5.29	5.78
Minimum	0.59	0.44	0.44	0.44	0.44	0.44
5% Quantile	1.31	1.00	0.95	0.90	0.90	0.90
50% Quantile	5.42	1.97	1.94	1.82	1.78	1.78
75% Quantile	5.98	7.95	9.16	2.71	2.14	2.09
90% Quantile	5.98	7.95	9.97	15.00	10.82	5.40
95% Quantile	5.98	7.95	9.97	15.00	20.00	25.00
99% Quantile	5.98	7.95	9.97	15.00	20.00	25.00
Maximum	6.00	8.00	10.00	15.00	20.00	25.00

6. Conclusions

- ES-hedging is a reasonable alternative to classical approaches (superhedging, mean-variance-hedging) for hedging contingent claims in *incomplete* markets.
- Calculating ES-strategies in discrete models is equivalent to the iterative solution of linear programs whose number increases *exponentially* with respect to the number of trading dates.
- Calculating LES-strategies in discrete models is equivalent to the iterative solution of linear programs whose number increases only *linearly* with respect to the number of trading dates.
- LES-strategies approximate ES-strategies quite accurately.
- ES and LES-hedging is flexible enough to consider additional constraints on the hedging costs.

The (discounted) Expected Shortfall of a risky position X defined through

$$\rho(X) = ESD(X) = E_P(\max(-X/B_T; 0)) \equiv E_P(X^-/B_T)$$

fulfills:

Axiom S: *(Subadditivity)* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Axiom PH: *(Positive homogeneity)* $\rho(\alpha \cdot X) = \alpha \cdot \rho(X)$ when $\alpha \geq 0$.

Axiom M: *(Monotonicity)* $\rho(Y) \leq \rho(X)$ when $X \leq Y$.

but not

Axiom T: *(Translation invariance)* $\rho(X + \alpha \cdot B_T) = \rho(X) - \alpha$.

$\rho(x) = ESD(X)$ fulfills instead

Axiom T': For all risky positions X and all real numbers α we have the inequality

$$-B_T^{-1} \cdot E_P(X + \alpha \cdot B_T) \leq \rho(X) - \alpha. \qquad \alpha \in IR.$$