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Continuous-Time Portfolio Optimization Under Terminal Wealth Constraints

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Abstract: Typically portfolio analysis is based on the expected utility or the mean-variance approach. Although the expected utility approach is the more general one, practitioners still appreciate the mean-variance approach. We give a common framework including both types of selection criteria as special cases by considering portfolio problems with terminal wealth constraints. Moreover, we propose a solution method for such constrained problems.

Key Words: Portfolio optimization, complete markets, martingale method, constraints.

1 Introduction

The continuous-time portfolio problem consists of maximizing total expected utility of consumption and/or of terminal wealth. This problem is usually formulated for a securities market with $d + 1$ assets where one of them is a default-free bond whose instantaneous rate of return $r(t)$ may (possibly randomly) fluctuate, and the other d securities are stocks whose prices have (randomly fluctuating) mean rates of return $b_i(t)$ and volatility coefficients $\sigma_{ij}(t)$, and are driven by independent Wiener processes.

In this paper we consider a continuous-time portfolio problem with constraints on the terminal wealth of an investor. Such constraints occur, for instance, if the traditional mean-variance approach of portfolio analysis is formulated in continuous time. The possibility to consider mean-variance problems (in continuous time) and continuous-time portfolio problems in a common framework is one of the attractive features of our constrained model, because the mean-variance approach is still of great practical importance. However, the mean-variance problem in continuous time has not been solved until recently (compare e.g. Duffie & Richardson (1991), Hipp (1993) or Schweizer (1993)). While Duffie & Richardson (1991) consider the hedging of a futures position, Hipp (1993) and Schweizer (1993) consider the hedging of general

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claims (including as a special case a mean-variance problem), but there will be slight differences to the mean-variance problem considered in this paper.

The paper is organized as follows. In section 2 we present the market model, its main characteristics, and a slight generalization of the usual martingale method to solve the (unconstrained) portfolio problem. Section 3 extends this method for constrained problems. There we present our main result which states existence of the constrained solution. In addition, this result offers a constructive way to find the optimal solution. As an application we formulate and solve a mean-variance problem in continuous time.

2 Solution of the Unconstrained Portfolio Problem

We start with the classical model in which an investor makes consumption and investment decisions continuously in time. He is endowed with an initial wealth of x and tries to maximize his utility from consumption over a fixed time interval $[0, T]$ and/or from terminal wealth in the time horizon T. The available $d + 1$ securities are a riskless bond and d risky stocks. Let $P_o(t)$ be the price of the bond and $P_i(t)$ be the price of stock i, $i = 1, \ldots, d$, at time t. These prices are governed by the equations

$$dP_o(t) = P_o(t)r(t)dt \qquad P_o(0) = 1 \tag{2.1}$$

$$dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t) \right], \qquad i = 1, \ldots, d, \qquad P_i(0) = p_i \tag{2.2}$$

where $W(t) = (W_1(t), \ldots, W_d(t))^T$ is a d-dimensional Brownian motion defined on a complete probability space (Ω, F, P). The informaton structure is given by the Brownian filtration $\{F_t\}_{t \in [0,T]}$, $T < \infty$. The market coefficients $r(t)$, $b(t) = (b_1(t), \ldots, b_d(t))^T$ and $\sigma(t) = (\sigma_{ij}(t))_{i,j=1(1)d}$ are all assumed to be progressively measurable and uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$. $\sigma(t)\sigma(t)^T$ is required to be uniformly positive definite. Further, we define the risk premium process

$$\theta(t) := \sigma(t)^{-1}(b(t) - r(t)\cdot \underline{1}) \qquad \forall t \in [0, T] \tag{2.3}$$

where $\underline{1} = (1, \ldots, 1)^T \in R^d$. Due to the assumptions on r, b, and σ, $\theta(t)$ is also progressively measurable and uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$. Let $X(t)$ be the wealth of an investor at time $t \in [0, T]$. As usual, we then define

Definition 2.1:

i) *A portfolio process is an* R^d – *valued, progressively measurable process* $\pi(t)$, $t \in [0, T]$, *with*

$$\int_0^T \|\pi(t)X(t)\|^2 dt < \infty \qquad a.s. \tag{2.4}$$

ii) *A consumption process is a non-negative, progressively measurable process* $c(t)$, $t \in [0, T]$, *with*

$$\int_0^T c(t)dt < \infty \qquad a.s. \tag{2.5}$$

Moreover, we require an investor to act self-financing. That means, his wealth only changes due to consumption or gains/losses from investment in the bond and the stocks. His wealth process $X(t)$ corresponding to a self-financing portfolio/consumption strategy (π, c) is given as the (unique) solution of the stochastic differential equation

$$\begin{aligned} dX(t) &= X(t)\{(1 - \pi(t)^T \underline{1})r(t)dt + \pi(t)^T[b(t)dt + \sigma(t)dW(t)]\} \\ &\qquad - c(t)dt \ , \qquad X(0) = x \end{aligned} \tag{2.6}$$

From now on we consider only investors who trade in such a way that they achieve a non-negative wealth over the whole time interval $[0, T]$. We therefore call a wealth process $X(t)$ corresponding to the strategy (π, c) and initial wealth $x > 0$ admissible if $X(t)$ solves equation (2.6), and if we have

$$X(t) \geq 0 \qquad \forall t \in [0, T] \qquad \text{a.s.} \ . \tag{2.7}$$

Let further

$$A(x) := \{(\pi, c)\colon X(t) \geq 0 \ \forall t \in [0, T] \text{ a.s.}, X(0) = x\} \tag{2.8}$$

be the set of admissible strategies (with initial wealth $x > 0$). The main characteristcs of the market model are summarized in the following proposition (see for example Cvitanic and Karatzas (1992)):

Proposition 2.1: Define the following deflator process

$$H(t) = exp\left(-\int_0^t (r(s) + 1/2\|\theta(s)\|^2)ds - \int_0^t \theta(s)dW(s)\right) , \qquad t \in [0, T] \ . \quad (2.9)$$

a) *For every* $(\pi, c) \in A(x)$ *with corresponding wealth process* $X(t)$ *we have*

$$E\left(\int_0^t H(s)c(s)ds + H(t)X(t)\right) \leq x \tag{2.10}$$

b) *For every consumption process* $c(t)$, $t \in [0, T]$, *and every non-negative,* F_T *– measurable random variable B with*

$$x := E\left(\int_0^T H(s)c(s)ds + H(T)B \right) < \infty \tag{2.11}$$

there exists a portfolio process $\pi(t)$, $t \in [0, T]$, *with corresponding wealth process* $X(t)$ *such that*

$$(\pi, c) \in A(x) \tag{2.12}$$

$$X(T) = B \qquad a.s. \tag{2.13}$$

We now give a short review of the so called martingale method of solving an unconstrained portfolio problem which is, for instance, presented in Karatzas (1989).

Definition 2.2: A strictly concave C^1*-function* $U: (0, \infty) \to R$ *with*

$$U'(0) := \lim_{c \downarrow 0} U'(c) > 0 \ , \qquad \text{where } U'(0) = +\infty \text{ is permitted} \tag{2.14}$$

$$\exists z \in R \cup \{+\infty\} \qquad \text{with } U'(z) = 0 \tag{2.15}$$

is called a (generalized) utility function.

Remark 2.1:

a) A set of functions $U(t, .)$, $t \in [0, T]$, will be also called a utility function if for every fixed $t \in [0, T]$, $U(t, .)$ is a utility function in the second variable, if $U(., x)$ is a continuous function in the first variable and if there exists a $z \in R \cup \{+\infty\}$ with $U'(t, z) = 0$ for every $t \in [0, T]$, where

$$U'(t, x) := \frac{\partial}{\partial x} U(t, x) , \qquad t \in [0, T] , \qquad x > 0 . \tag{2.16}$$

b) Definition 2.2 implies that U' is strictly decreasing on $[0, z]$ with U': $[0, z] \to [0, U'(0)]$ and has a strictly decreasing, continuous inverse function I^*: $[0, U'(0)] \to [0, z]$. With the notation

$$I(y) := \begin{cases} I^*(y) , & y \in [0, U'(0)] \\ 0 , & y \geq U'(0) \end{cases} \tag{2.17}$$

we have

$$U(I(y)) \geq U(c) + y(I(y) - c) , \qquad y \in (0, \infty) , \qquad c \geq 0 \tag{2.18}$$

where $U(0)$ is defined as $U(0) := \lim_{c \downarrow 0} U(c)$. (2.18) remains valid for $y = 0$ if we have $z = +\infty$.

Let $X^{x,\pi,c}(t)$, $t \in [0, T]$, be the wealth process of an investor who starts with an initial wealth of $x > 0$ and who uses an admissible strategy (π, c). The investor's utility of using this strategy is defined as

$$J(x; \pi, c) := E\left(\int_0^T U_1(t, c(t))dt + U_2(X^{x,\pi,c}(T)) \right) \tag{2.19}$$

where U_1, U_2 are utility functions in the sense of definition 2.2 or remark 2.1 a), respectively.

Definition 2.3: The unconstrained portfolio problem (of an investor with initial wealth $x > 0$) is the optimization problem

$$\sup_{(\pi, c) \in A'(y), y \leq x} J(y; \pi, c)$$

where (2.20)

$$A'(y) := \left\{ (\pi, c) \in A(y) : E\left(\int_0^T U_1(t, c(t))^- dt + U_2(X^{x,\pi,c}(T))^- \right) < \infty \right\}$$

Remark 2.2:

a) If $U_1(t, x)$, $U_2(x)$ are strictly increasing in x then there is no need for the inequality "$y \leq x$" in the optimization problem, and we can then confine ourselves to strategies $(\pi, c) \in A'(x)$.
b) The restriction for the admissible strategies in the definition of $A'(y)$ can be interpreted that it is not prohibited to get an infinite utility from consumption/terminal wealth.

As in Karatzas (1989) we define the function $X: (0, \infty) \to R$ by

$$X(\lambda) := E\left[I_2(\lambda H(T))H(T) + \int_0^T H(t)I_1(t, \lambda H(t))dt \right] \qquad \text{for } \lambda > 0 \tag{2.21}$$

where I_1, I_2 are the "inverses" of U_1, U_2 in the sense of (2.17). The main characteristics of $X(\lambda)$ are summarized in the following proposition (compare Cvitanic and Karatzas (1992)).

Proposition 2.2: Assume

$$X(\lambda) < \infty \qquad \forall \lambda \in (0, \infty) \ , \tag{2.22}$$

and in the case of

$$U_1'(t, 0) < \infty \qquad \forall t \in [0, T] \qquad \textit{and} \quad U_2'(0) < \infty \tag{2.23}$$

assume further that $\theta(t)$, $t \in [0, T]$, *is deterministic with*

$$\int_0^t \|\theta(s)^2\| \, ds > 0 \qquad \forall t \in [0, T] \tag{2.24}$$

Then X *is continuous on* $(0, \infty)$, *strictly decreasing with*

$$X(\infty) := \lim_{\lambda \to \infty} X(\lambda) = 0 \tag{2.25}$$

$$X(0) := \lim_{\lambda \to 0} X(\lambda) = \begin{cases} \infty \ , & \text{if } \lim\limits_{z \to \infty} U_2'(z) = 0 \text{ or } \lim\limits_{z \to \infty} U_1'(t, z) = 0 \ \forall t \in [0, T] \\ z_1 E \int_0^T H(t)dt + z_2 EH(T) \ , & \text{else} \end{cases} \tag{2.26}$$

where z_1, z_2 are the values with $U_1'(t, z_1) = 0$ $\forall t \in [0, T]$ and $U_2'(z_2) = 0$, respectively.

Remark 2.3: The assumption (2.24) is that of a deterministic "mean-variance trade off". It is an open question if this assumption can be relaxed in the continuous-time case (compare Schweizer (1993)).

Proof:

a) The continuity of X follows from the continuity of $I_1(t, .)$ and I_2 by the dominated convergence theorem.
b) (compare Karatzas, Lehoczky and Shreve (1987)) $I_1(t, .)$ is strictly decreasing on $(0, U_1'(t, 0))$. If we can show that

$$P(\lambda H(t) < U_1'(t, 0) \text{ for some } t \in [0, T]) > 0 \tag{2.27}$$

for every fixed $\lambda \in (0, \infty)$ then it follows that

$$X_1(\lambda) := E \int_0^T H(t) I_1(t, \lambda H(t)) dt \tag{2.28}$$

is strictly decreasing in $\lambda \in (0, \infty)$ because $H(t)I_1(t, \lambda H(t))$ is strictly decreasing in λ on the set that is characterized in (2.27) and identically zero on its complement. Since $H(t)$ and $U_1'(t, 0)$ are assumed to be continuous, (2.27) also implies

$$P(\exists 0 \leq t_1 < t_2 \leq T \colon \lambda H(t) < U_1'(t, 0) \forall t \in (t_1, t_2)) > 0 \ , \tag{2.29}$$

and we get the claimed monotonicity of X. But we have

$$\begin{aligned} \ln(H(t)) &= -\int_0^t \theta(s)^T dW(s) - 1/2 \int_0^t \|\theta(s)\|^2 ds - \int_0^t r(s) ds \\ &= \tilde{W}_{A(t)} - \int_0^t (r(s) + 1/2 \|\theta(s)\|^2) ds \qquad \text{a.s.} \end{aligned} \tag{2.30}$$

where $\tilde{W}_{A(t)}$, $t \in [0, T]$, is a one dimensional Brownian motion with

$$A(t) = \int_0^t \|\theta(s)\|^2 ds \ , \qquad t \in [0, T] \ , \tag{2.31}$$

(compare Karatzas and Shreve (1987), p. 174 ff. "random time change"). Because $r(t)$, $\theta(t)$, $t \in [0, T]$, are bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$, there exists a real number K with

$$\int_0^t (r(s) + 1/2\|\theta(s)\|^2)ds \leq K \qquad \forall t \in [0, T] \tag{2.32}$$

Because $A(t)$ is a positive deterministic function, the characteristics of Brownian motion imply that

$$P(\ln(H(t)) < u \text{ for some } t \in [0, T]) > 0 \qquad \text{for } u > 0\ (u \text{ arbitrary})\ . \tag{2.33}$$

Hence (2.27) is proved. Analogous considerations for

$$X_2(\lambda) := E(H(T)I_2(\lambda H(T)) \tag{2.34}$$

imply

$$P(\lambda H(T) < U_2'(0)) > 0\ . \tag{2.35}$$

Thus, X_2 and also $X(\lambda) := X_1(\lambda) + X_2(\lambda)$ are strictly decreasing in $\lambda \in (0, \infty)$, too.

Notice further that (2.27) or (2.35) are always fulfilled if $U_1'(t, 0) = \infty$ for all $t \in [0, T]$ or $U_2'(0) = \infty$.

c) $I_1(t, \infty) = I_2(\infty) = 0$ ($\forall t \in [0, T]$) imply (2.25) by the monotone convergence theorem.

d) First consider the case

$$\lim_{z \to \infty} U_2'(z) = 0 \tag{2.36}$$

Because I_1 and I_2 are non-negative functions, Fatou's lemma implies

$$\liminf_{\lambda \to 0} X(\lambda) \geq X_2(\lambda) \geq E\left(H(T) \liminf_{\lambda \to 0} I_2(\lambda H(T))\right) = \infty\ . \tag{2.37}$$

For the same reasons we have in the case of

$$\lim_{z \to \infty} U_1'(t, z) = 0\ \forall t \in [0, T] \tag{2.38}$$

$$\liminf_{\lambda\to 0} X(\lambda) \geq \liminf_{\lambda\to 0} X_1(\lambda) \geq E\left(\int_0^T H(t) \liminf_{\lambda\to 0} I_1(t, \lambda H(t))dt\right) = \infty \ . \tag{2.39}$$

If (2.36) or (2.38) are not satisfied, then we have

$$\limsup_{\lambda\to 0} X(\lambda) \leq z_1 E \int_0^T H(t)dt + z_2 EH(T) \ , \tag{2.40}$$

because of $I_1(t, x) \leq z_1$ and $I_2(x) \leq z_2$ for every (t, x) from the regions where I_1 and I_2 are defined. On the other hand, Fatou's lemma implies

$$\liminf_{\lambda\to 0} X(\lambda) \geq E\left(\int_0^T H(t) \liminf_{\lambda\to 0} I_1(t, \lambda H(t))dt + H(T) \liminf_{\lambda\to 0} I_2(\lambda H(T))\right)$$

$$= z_1 E \int_0^T H(t)dt + z_2 EH(T) \ . \tag{2.41}$$

Hence, the proof is complete. □

If we define

$$z^* := \begin{cases} z_1 E \int_0^T H(t)dt + z_2 EH(T) \ , & \text{if } z_1 \text{ and } z_2 \text{ are finite} \\ \infty \ , & \text{else} \end{cases} \tag{2.42}$$

and $X(\infty)$ and $X(0)$ as in (2.25) and (2.26) then there exists a continuous and strictly decreasing inverse function Y of X on $[0, \infty]$ with

$$Y\colon [0, z^*] \to [0, \infty] \ . \tag{2.43}$$

Now we can give the solution to the unconstrained portfolio problem.

Proposition 2.3: Let $X > 0$, *assume* (2.22) *and in the case of* (2.23) *assume further* (2.24). *Then the optimal terminal wealth* ξ *and the optimal consumption process* $c_o(t)$, $t \in [0, T]$, *are given by*

$$\xi := \begin{cases} z_2 \ , & \text{if } x \geq z^* \\ I_2(Y(x)H(T)) \ , & \text{else} \end{cases} \tag{2.44}$$

$$c_o(t) := \begin{cases} z_1 , & \text{if } x \geq z^* \\ I_1(t, Y(x)H(t)) , & \text{else} \end{cases} \tag{2.45}$$

and there exists a $y_o \in [0, x]$ and a portfolio process $\pi_o(t)$, $t \in [0, T]$, such that we have

$$(\pi_o, c_o) \in A'(y_o) , \qquad X^{y_o, \pi_o, c_o}(T) = \xi \qquad a.s.$$

and (2.46)

$$J(y, \pi_o, c_o) = \sup_{(\pi, c) \in A'(y), y \leq x} J(y; \pi, c)$$

i.e. (π_o, c_o) is a solution to the unconstrained portfolio problem.

Proof:

Case 1: $x \geq z^*$
Because $U_1(t, .)$ and $U_2(.)$ attain their absolute maxima for $z_1 = I_1(t, 0)$ for every $t \in [0, T]$ and for $z_2 = I_2(0)$, respectively, we have

$$\int_0^T U_1(t, I_1(t, 0))dt + U_2(I_2(0)) \geq \int_0^T U_1(t, c(t))dt + U_2(X^{y,\pi,c}(T)) \qquad \text{f.s.} \tag{2.47}$$

for every $(\pi, c) \in A'(y)$ with $y \leq x$. Thus, the choices of ξ and $c_o(t)$ are in this case pathwise optimal and therefore also optimal for the unconstrained portfolio problem. The existence of a portfolio process $\pi_o(t)$ with $(\pi_o, c_o) \in A(z^*)$ and $X^{z^*,\pi,c}(T) = \xi$ a.s. follows from Proposition 2.1. Notice further that $c_o(t)$ and ξ are deterministic, so we have

$$E\left(\int_0^T U_1(t, c_o(t))^- dt + U_2(X^{z^*,\pi,c}(T))^-\right) = \int_0^T U_1(t, z_1)^- dt + U_2(z_2)^- < \infty , \tag{2.48}$$

which implies $(\pi_o, c_o) \in A'(z^*)$. Hence, everything is proved in the case $x \geq z^*$.
Case 2: $x < z^*$: compare Cvitanic and Karatzas (1992), Theorem 7.4. □

Remark 2.4:

a) If there exist finite values z_1 and z_2 with $U_1'(t, z_1) = 0 \ \forall t \in [0, T]$ and $U_2'(z_2) = 0$ then Proposition 2.3 especially implies that the choice of "optimal consumption" $c(t) = z_1 \ \forall t \in [0, T]$ and the choice of "optimal terminal wealth" $\xi = z_2$ are optimal for the unconstrained portfolio problem only if these choices are financiable (this means that the initial capital x of the investor exceeds z^*). If only one of the two values z_1, z_2 is finite, then Proposition 2.3 states that it is never optimal to choose this value for the consumption rate or the terminal wealth, respectively.
b) Explicit calculations for logarithmic utility can be found in Karatzas (1989), while HARA-functions are dealt with in Merton (1971).

Example: Minimal deviation from a target value.
Let

$$U_1(t, .) \equiv 0 \ \forall t \in [0, T] \ , \quad U_2(x) = -1/2(x - K)^2 \quad \forall x > 0 \ , \tag{2.49}$$

where $K > 0$ is a given constant ("the target value"). The optimization problem

$$\inf_{(\pi,c) \in A'(y),\, y \le x} 1/2E(X^{y,\pi,c}(T) - K)^2 \tag{2.50}$$

is equivalent to the unconstrained portfolio problem

$$\sup_{(\pi,c) \in A'(y),\, y \le x} EU_2(X^{y,\pi,c}(T)) \tag{2.51}$$

We have to distinguish two cases:

a) $x \ge K \cdot EH(T)$
Proposition 2.3 implies:

$$\xi := K \qquad \text{is the optimal attainable terminal wealth .} \tag{2.52}$$

b) $x < K \cdot EH(T)$
Proposition 2.3 implies:

$$\xi := I_2(Y(x)H(T)) = (K - Y(x)H(T)) \cdot 1_{\{H(T) \le (K/\lambda)\}} \tag{2.53}$$

is the optimal terminal wealth. Unfortunately

$$\begin{aligned} X(\lambda) &= E(H(T)(K - \lambda H(T))1_{\{H(T) \le (K/\lambda)\}}) \\ &= KE(H(T)1_{\{H(T) \le (K/\lambda)\}}) - \lambda E(H(T)^2 1_{\{H(T) \le (K/\lambda)\}}) \end{aligned} \tag{2.54}$$

cannot be solved explicitly for λ to obtain the inverse function Y. However, we can obtain a numerical solution in the case of constant coefficients. Consider especially the case $d = 1$. Then we have

$$\begin{aligned} X(\lambda) = K \exp(-rT)\phi &\left[\frac{rT - (1/2)\theta^2 T + \ln\left(\frac{K}{\lambda}\right)}{\theta\sqrt{T}}\right] \\ - \lambda \exp((-2r + \theta^2)T)\phi &\left[\frac{rT - (3/2)\theta^2 T + \ln\left(\frac{K}{\lambda}\right)}{\theta\sqrt{T}}\right] \end{aligned} \tag{2.55}$$

and we get $Y(x)$ by solving the non-linear equation $X(\lambda) = x$ which has a unique solution due to Proposition 2.2.

Remark 2.5: The above example shows a difference to the results of Schweizer (1993) who obtained a closed form solution for a slightly different mean-variance problem by projection techniques. His optimal terminal wealth is given by

$$\xi := K - (K - xe^{rT}) \exp(-\theta W_T - (3/2)\theta^2 T) \tag{2.56}$$

which can be negative with positive probability (for small x or small T). Hence, it is not an admissible terminal wealth for our type of portfolio problems (compare definition (2.8)). However, this is only a special case of the results of Schweizer (1993).

3 A Dual Method to Solve Portfolio Problems with Constraints on the Terminal Wealth

In this section we consider constrained portfolio problems of the following type:

$$\sup_{(\pi,c)\in A_c'(y),\, y\le x} J(y,\pi,c)$$

subject to (3.1)

$$EG_i(X^{y,\pi,c}(T)) \le 0 \ , \qquad i = 1,\ldots,k$$

where

$$G(u) = (G_1(u),\ldots,G_k(u))^T \ , \qquad k\in N \ , \qquad u>0 \tag{3.2}$$

is an R^k – valued function such that

$$U_2 - d^T G \qquad \text{is a utility function for all } d\in[0,\infty)^k \ . \tag{3.3}$$

Remark 3.1:

a) Condition (3.3) is always fulfilled if for all functions G_i, $i = 1, \ldots, k$, their negative analogues $(-G_i)$ are utility functions.
b) Strategies (π, c) having a terminal wealth $X^{y,\pi,c}(T)$ with

$$EG_i(X^{y,\pi,c}(T)) = -\infty \tag{3.4}$$

for some $i\in\{1,\ldots,k\}$ should also be called admissible, if they do not satisfy

$$EG_i(X^{y,\pi,c}(T))^+ = +\infty \tag{3.5}$$

for some $i\in\{1,\ldots,k\}$. So let

$$A_c'(y) := A'(y)\backslash\{(\pi,c)\in A'(y)\colon EG_i(X^{y,\pi,c}(T))^+ = \infty \text{ for some } i\in\{1,\ldots,k\}\} \tag{3.6}$$

be the set of admissible strategies for the constrained optimization problem (3.1).

The solution method for problem (3.1) is based on a modification of the corresponding method from deterministic optimization which is based on the saddle-point theorem (Compare Fletcher (1981) for the deterministic case). Let x be the fixed initial wealth of the investor. We define a stochastic analogue to the Lagrangian for problem (3.1) by

$$L((\pi, c), d) := E\left[\int_0^T U_1(t, c(t))dt + (U_2 - d^T G)(X^{y,\pi,c}(T))\right] \tag{3.7}$$

for $(\pi, c) \in A_c'(y)$, $y \leq x$, $d \in [0, \infty)^k$. Then we have

$$\inf_{d \geq 0} L((\pi, c), d) = \begin{cases} -\infty \ , & \text{if } EG(X^{y,\pi,c}(T)) \not\leq 0 \text{ (component wise)} \\ J(y, \pi, c) \ , & \text{else} \end{cases} \tag{3.8}$$

i.e. the optimization problem (3.1) is equivalent to the problem

$$\sup_{(\pi,c) \in A_c'(y), y \leq x} \ \inf_{d \geq 0} L((\pi, c), d) . \tag{3.9}$$

The obvious relation

$$\sup_{(\pi,c) \in A_c'(y), y \leq x} \ \inf_{d \geq 0} L((\pi, c), d) \leq \sup_{(\pi,c) \in A_c'(y), y \leq x} L((\pi, c), d^*) \tag{3.10}$$

for every $d \in [0, \infty)^k$ implies

$$\sup_{(\pi,c) \in A_c'(y), y \leq x} \ \inf_{d \geq 0} L((\pi, c), d) \leq \inf_{d \geq 0} \ \sup_{(\pi,c) \in A_c'(y), y \leq x} L((\pi, c), d) \tag{3.11}$$

Referring to the notations

$$\varphi(d) := \sup_{(\pi,c) \in A_c'(y), y \leq x} L((\pi, c), d) \ , \qquad d \in [0, \infty)^k \tag{3.12}$$

$$\psi(\pi, c) := \inf_{d \geq 0} L((\pi, c), d) \ , \qquad (\pi, c) \in A_c'(y) \ , \qquad y \leq x \tag{3.13}$$

we only have to find a pair $((\pi^*, c^*), d^*) \in A_c'(y) \times [0, \infty)^k$ for some $y \leq x$ with

$$\varphi(d^*) = \psi(\pi^*, c^*) \tag{3.14}$$

to prove equality in (3.11). This will be done in the proof of the following theorem.

Theorem 3.1: Let

$$G_i\colon [0, \infty) \to R \ , \qquad i = 1, \ldots, k \tag{3.15}$$

be convex functions and let the assumptions of Proposition 3.2 *be satisfied for every unconstrained portfolio problem with utility functions given by* (3.7). *Moreover, every solution* (π, c) *of these problems should be in* $A_c'(y)$.

a) *Assume that there exists a strategy* $(\pi_o, c_o) \in A_c'(y)$, $y \leq x$ *with*

$$EG(X^{y,\pi_o,c_o}(T)) < 0 \qquad (\textit{component wise}) \ , \tag{3.16}$$

and that problem (3.1) *has a finite optimal solution. Then we have*

$$\sup_{(\pi,c) \in A_c'(y), y \leq x} \ \inf_{d \geq 0} L((\pi, c), d) = \inf_{d \geq 0} \ \sup_{(\pi,c) \in A_c'(y), y \leq x} L((\pi, c), d) \ , \tag{3.17}$$

and there exists a pair $((\pi^*, c^*), d^*) \in A_c'(y) \times [0, \infty)^k$ *for a* $y \leq x$ *with*

$$\varphi(d^*) = \psi(\pi^*, c^*) \tag{3.18}$$

$$0 = d^{*T} EG(X^{y,\pi^*,c^*}(T)) \ . \tag{3.19}$$

b) *If the dual problem has no finite optimum, i.e.*

$$\inf_{d \geq 0} \varphi(d) = -\infty \ , \tag{3.20}$$

then there is no strategy $(\pi, c) \in A_c'(y)$ *for any* $y \leq x$ *with* $EG_i(X^{y,\pi,c}(T)) \leq 0$, $i = 1, \ldots, k$.

c) *If the primal problem* (3.1) *has no finite optimum, i.e.*

$$\sup_{(\pi,c) \in A_c'(y), y \leq x} \psi(\pi, c) = +\infty \ , \tag{3.21}$$

then there is no admissible solution for the dual problem, i.e. there is no $d \in [0, \infty)^k$ *with*

$$\varphi(d) < +\infty \tag{3.22}$$

Proof:

a) We only have to show the existence of the pair $((\pi^*, c^*), d^*)$. Then, (3.17) follows from (3.14) and (3.11), and (3.19) will be implied by (3.8). Let

$$K_1 := \{(u, z) \in R \times R^k : \exists (\pi, c) \in A'_c(y) \text{ for some } y \leq x \text{ with}$$

$$u \leq J(y, \pi, c),\ EG(X^{y,\pi,c}(T)) \leq z\} \tag{3.23}$$

$$K_2 := \left\{(u, z) \in R \times R^k : u \geq \sup_{\substack{(\pi,c) \in A'_c(y),\, y \leq x \\ EG(X^{y,\pi,c}(T)) \leq 0}} J(y, \pi, c),\ z \leq 0\right\} \tag{3.24}$$

The assumptions of the theorem imply that K_2 is a non-empty convex set. Next we show:

$$K_1 \text{ is (also) a non-empty convex set} \tag{3.25}$$

$K_1 \neq \emptyset$ follows directly from the assumptions of a). So it remains to prove the convexity of K_1. Let $(a, b), (c, d) \in K_1$. Then there exist

$$(\pi_1, c_1) \in A'_c(y_1) \text{ for some } y_1 \leq x \text{ with } a \leq J(y_1, \pi_1, c_1) \text{ and}$$

$$EG(X^{y_1,\pi_1,c_1}(T)) \leq b \tag{3.26}$$

$$(\pi_2, c_2) \in A'_c(y_2) \text{ for some } y_2 \leq x \text{ with } c \leq J(y_2, \pi_2, c_2) \text{ and}$$

$$EG(X^{y_2,\pi_2,c_2}(T)) \leq d \tag{3.27}$$

Defining

$$Y := \lambda X^{y_1,\pi_1,c_1}(T) + (1-\lambda) X^{y_2,\pi_2,c_2}(T) \qquad \text{for } \lambda \in [0, 1] \tag{3.28}$$

$$c(t) := \lambda c_1(t) + (1-\lambda) c_2(t) \qquad \text{for } \lambda \in [0, 1]\ , \tag{3.29}$$

we get

$$x^* = E\left(\int_0^T H(t)c(t)dt + H(T)Y\right)$$

$$= \lambda E\left(\int_0^T H(t)c_1(t)dt + H(T)X^{y_1,\pi_1,c_1}(T)\right)$$

$$+ (1-\lambda)E\left(\int_0^T H(t)c_2(t)dt + H(T)X^{y_2,\pi_2,c_2}(T)\right)$$

$$\le \lambda y_1 + (1-\lambda)y_2$$

$$\le x\ , \tag{3.30}$$

where the first inequality follows from (3.26)) and (3.27). Proposition 2.1 implies the existence of a portfolio process $\pi(t)$, $t \in [0, T]$, with

$$(\pi, c) \in A_c(x^*) \qquad \text{and} \qquad Y = X^{x^*,\pi,c}(T) \qquad \text{a.s.}\ , \tag{3.31}$$

and the concavity of U_1 and U_2 together with (3.28), (3.29) imply

$$J(x^*, \pi, c) \ge \lambda J(y_1, \pi_1, c_1) + (1-\lambda)J(y_2, \pi_2, c_2) \ge \lambda a + (1-\lambda)c \tag{3.32}$$

We further have (remind the convexity of the G_i, $i = 1, \ldots, k$):

$$\begin{aligned} EG(X^{x^*,\pi,c}(T)) &= EG(Y) \\ &\le \lambda EG(X^{y_1,\pi_1,c_1}(T)) + (1-\lambda)EG(X^{y_2,\pi_2,c_2}(T)) \\ &\le \lambda b + (1-\lambda)d \end{aligned} \tag{3.33}$$

Thus (3.32) and (3.33) imply

$$\lambda(a, b) + (1-\lambda)(c, d) \in K_1 \qquad \forall \lambda \in [0, 1]\ , \tag{3.34}$$

i.e. the convexity of K_1. The assumption of a finite optimal solution to the constrained problem (3.1) shows that we have

$$\mathring{K}_2 \neq \emptyset \tag{3.35}$$

(where $\mathring{K}_2$ is the interior of K_2) and the definitions of K_1 and K_2 give

$$K_1 \cap \mathring{K}_2 = \emptyset \tag{3.36}$$

The separation theorem for convex sets (see e.g. Ioffe & Tichomiroff (1979)) implies the existence of a functional $w^* \neq 0$ with $w^* = (w_1^*, w_2^{*T})^T \in$

$R \times R^k$ and

$$(w^*)^T x \le (w^*)^T y \qquad \forall x \in K_1 \;, \qquad y \in K_2 \;, \tag{3.37}$$

especially

$$\sup_{x \in K_1} (w^*)^T x \le \inf_{y \in K_2} (w^*)^T y \tag{3.38}$$

As a consequence of (3.38) and the form of K_2, $w^* = (w_1^*, w_2^{*T})^T$ has to satisfy

$$w_1^* \in [0, \infty) \;, \qquad w_2^* \le 0 \qquad \text{(component wise)} \tag{3.39}$$

To prove this, choose y^* as the maximum of the objective function subject to the contraints in (3.1) (which exists due to our assumptions). Then $(y^*, z) \in K_2$ satisfies $z \le 0$, but there exist pairs (y^*, z) in K_1 with $z > 0$. Now (3.38) implies

$$w_2^* \le 0 \;. \tag{3.40}$$

So (3.38) and the existence of pairs of the form $(u, 0)$ which are elements of K_1 and K_2 result in

$$w_1^* \ge 0 \;. \tag{3.41}$$

We can even show

$$w_1^* > 0 \;, \tag{3.42}$$

because the assumption $w_1^* = 0$, inequality (3.38), and the fact that we have $(y^*, 0) \in K_2$ imply

$$(w^*)^T \begin{pmatrix} x \\ z \end{pmatrix} = w_2^* z \le 0 \qquad \forall (y, z^T) \in K_1 \;. \tag{3.43}$$

This is a contradiction to the existence of the pair (π_o, c_o) with property (3.16), because the assumption $w_1^* = 0$ and relation (3.40) imply that there exist some negative components of w_2^* (remember $w^* \neq 0$), so this pair doesn't satisfy (3.43). Hence, we have (3.42).

Because of relation (3.42) we can assume $w_1^* = 1$ w.l.o.g. If we choose y^* as the maximum of the objective function subject to the constraints in (3.1) we have

$$y^* = \sup_{\substack{(\pi,c) \in A_c'(y),\, y \le x \\ EG(X^{y,\pi,c}(T)) \le 0}} J(y, \pi, c) = \sup_{(\pi,c) \in A_c'(y),\, y \le x} \inf_{d \ge 0} L((\pi, c), d) \tag{3.44}$$

and on the other hand

$$(y^*, 0) \in K_1 \cap K_2 \ . \tag{3.45}$$

Thus, because of (3.38) and (3.45) we have

$$\sup_{(x,z) \in K_1} (w^*)^T \begin{pmatrix} x \\ z \end{pmatrix} = y^* = \inf_{(x,z) \in K_2} (w^*)^T \begin{pmatrix} x \\ z \end{pmatrix} \tag{3.46}$$

Furthermore, we have

$$\begin{aligned} y^* &= \sup_{(u,z) \in K_1} (w_1^*, w_2^{*T})^T \begin{pmatrix} u \\ z \end{pmatrix} = \sup_{(u,z) \in K_1} (u + w_2^{*T} z) \\ &\ge \sup_{(\pi,c) \in A_c'(y),\, y \le x} (J(y, \pi, c) + w_2^{*T} EG(X^{y,\pi,c}(T))) \\ &\ge \sup_{\substack{(\pi,c) \in A_c'(y),\, y \le x \\ EG(X^{y,\pi,c}(T)) \le 0}} J(y, \pi, c) = y^* \ , \end{aligned} \tag{3.47}$$

where the first inequality follows from the fact that the pairs $(J(y, \pi, c), EG(X^{y,\pi,c}(T)))$ are also in K_1. Relation (3.47) and the definition

$$d^* := -w_2^* \ , \tag{3.48}$$

imply the inequality

$$\begin{aligned} \inf_{d \ge 0} \sup_{(\pi,c) \in A_c'(y),\, y \le x} L((\pi, c), d) &\le \varphi(d^*) = y^* \\ &= \sup_{(\pi,c) \in A_c'(y),\, y \le x} \inf_{d \ge 0} L((\pi, c), d) \ . \end{aligned} \tag{3.49}$$

Hence, equality (3.17) is proved (recall that the opposite inequality to (3.49) is always valid). The existence of the strategy (π^*, c^*) fulfilling equality (3.18) follows from Proposition 2.3, because it solves the unconstrained portfolio problem

$$\sup_{(\pi,c)\in A_c'(y),\, y\le x} L((\pi, c), d^*) \ . \tag{3.50}$$

b) Relations (3.20) and (3.11) imply

$$-\infty = \inf_{d\ge 0} \varphi(d) \ge \sup_{(\pi,c)\in A_c'(y),\, y\le x} \inf_{d\ge 0} L((\pi, c), d) \ge \inf_{d\ge 0} L((\pi, c), d) \tag{3.51}$$

for every $(\pi, c) \in A_c'(y)$, $y \le x$. Combining this with equality (3.8), we can conclude that there is no strategy $(\pi, c) \in A_c'(y)$, $y \le x$, with $EG(X^{y,\pi,c}(T)) \le 0$.

c) Relations (3.11) and (3.21) imply statement c) in an analogous way to the proof of b). □

Remark 3.2: The theorem offers a separation of the solution of the constrained optimization problem (3.1) into two steps:

Step 1: Solve the unconstrained portfolio problem

$$\sup_{(\pi,c)\in A_c'(y),\, y\le x} L((\pi, c), d) \qquad (d \in [0, \infty)^k \text{ arbitrary but fixed}) \tag{3.52}$$

Step 2: Minimize the solution $L((\pi^*(d), c^*(d)), d)$ of Step 1 with respect to $d \in [0, \infty)^k$.

Example 3.1: A mean-variance problem

Consider a continuous-time analogue of the traditional mean-variance problem:

$$\inf_{(\pi,0)\in A_c'(y),\, y\le x} \mathrm{Var}(X^{y,\pi}(T)) \tag{3.53}$$

subject to $E(X^{y,\pi}(T)) \geq K$

where $K > 0$ is a given constant and $x > 0$ is the initial wealth of the investor.
First, we have to consider two different cases to convert

$$\operatorname{Var}(X^{y,\pi}(T)) = E((X^{y,\pi}(T) - EX^{y,\pi}(T))^2) \tag{3.54}$$

into the form of a utility function in the sense of definition 2.2.

Case 1: $r \geq \frac{1}{T}\ln\left(\frac{K}{x}\right)$

Here, the terminal wealth of K can be reached with a zero variance by a pure bond strategies.

Case 2: $r < \frac{1}{T}\ln\left(\frac{K}{x}\right)$ and $b \neq r$

Notice that every strategy that satisfies the expectation constraint must now include stock investment. Therefore, the unconstrained minimum variance of zero cannot be attained. Hence, the expectation constraint must be satisfied as an equality for an optimal solution. So it is easy to show that problem (3.53) is equivalent to the following auxiliary problem

$$\sup_{(\pi,0)\in A_c'(y),\, y\leq x} \quad -(1/2)E(X^{y,\pi}(T) - K)^2 \tag{3.55}$$

subject to $K - EX^{y,\pi}(T) \leq 0$

Therefore, we are able to solve the problem with the help of Theorem 3.1:
Fix $d \geq 0$, solve the unconstrained problem

$$\sup_{(\pi,0)\in A_c'(y),\, y\leq x} \quad -E((1/2)(X^{y,\pi}(T) - K)^2 + d(K - X^{y,\pi}(T))$$

$$= \sup_{(\pi,0)\in A_c'(y),\, y\leq x} \quad -E((1/2)X^{y,\pi}(T)^2 - (K + d)X^{y,\pi}(T) + (1/2)K^2 + dK)$$

$$= \left[\sup_{(\pi,0)\in A_c'(y),\, y\leq x} \quad -E((1/2)(X^{y,\pi}(T) - (K + d))^2)\right] + (1/2)d^2 \;, \tag{3.56}$$

and then minimize the solution in $d \in [0, \infty)$. Notice that the optimization problem in the last line of relation (3.56) is an unconstrained problem of the type considered in our example in section 2. The remark about its (possible numerical) solution is here valid too. Furthermore, the Lagrange multiplier d figures out the auxiliary problem of the form (3.55) with the correct target value $K + d$. The solution of this auxiliary problem delivers the solution of the mean-variance problem.

In addition, we give some numerical results for case 2. Let the initial wealth of the investor be $x = 100$, the bond interest rate $r = 0.1$. We let the mean rate of stock return take the values $b = 0.15$ or $b = 0.25$, for the volatility we choose $\sigma = 0.05$, and we take $T = 1$ or $T = 10$. For all the possible combinations of these values we consider three values for the desired minimal expected terminal wealth K. The lowest of these is always a little bit above the terminal wealth attained by the pure bond strategy, whereas the highest value is approximately the expected terminal wealth achieved by the pure stock strategy. The results are summarized in tables 1 to 4.

Table 1. $T = 1, b = 0.15$

K	111	113.5	116
Variance	0.14	5.18	17.55
d	0.56	3.47	6.43

Table 2. $T = 1, b = 0.25$

K	111	120	128
Variance	$3 \cdot 10^{-5}$	0.04	0.20
d	$2 \cdot 10^{-5}$	0.01	0.03

Table 3. $T = 10, b = 0.15$

K	275	360	445
Variance	$9 \cdot 10^{-4}$	4.91	39.63
d	$5 \cdot 10^{-4}$	0.17	0.72

Table 4. $T = 10, b = 0.25$

K	275	790	1200
Variance	≈ 0	≈ 0	≈ 0
d	≈ 0	≈ 0	≈ 0

Table 5. $T = 10$, $b = 0.25$, $\sigma = 0.2$

K	275	790	1200
Variance	0.04	1215	8006
d	0.02	130	790

Clearly, an increasing target value leads to an increasing variance and a higher value of the Lagrange multiplier d. Furthermore, comparing Table 1 to 2 and Table 3 to 4, respectively, we notice that the corresponding target values can be reached (in the mean) with a smaller variance in the case of a higher mean rate of stock return. Maybe the most remarkable results are the small values of the minimum variances. Compare for example the low variance of 17.55 in the last column of Table 1 with the variance of the pure stock strategy which is approximately 34. More extreme examples are the very small entries in Tables 2, 3, and 4. These values can be explained by two facts. Due to our choice of the market coefficients it is very advantageous to invest in the stock (and to sell bonds short!). This stock investment results in gains that are with a high probability greater than that obtained from bond investment. Immediate transfer of these gains to bond positions leads to a reduction of the variance of the terminal wealth. In the cases when $T = 10$ this process of obtaining high gains from stock investment and putting them immediately into bond positions runs a longer time. This results in a greater reduction of variance. This is not the case when the risk premium $(b - r)/\sigma$ is small. It is also clear that the minimum variance of the terminal wealth increases with increasing stock volatility. For illustration, Table 5 shows how the entries of Table 4 will change when $\sigma = 0.05$ is replaced by $\sigma = 0.20$.

This increase of variance is not so tremendous as it seems at a first glance. The variance of the pure stock strategy (resulting in an expected terminal wealth of approximately 1200) is namely nearly 729932 which is 90 times higher than the minimum variance for $K = 1200$.

Notice that these results differ from the formulae in Schweizer (1993) because we only consider strategies leading to a non-negative terminal wealth.

4 Conclusion

We consider a portfolio problem with constraints on the terminal wealth of an investor. As a result, we give a theorem about the existence of solutions for a very general class of such constrained problems. This theorem also suggests a solution method for the problem benefitting of existing methods for solving unconstrained portfolio problems. As a consequence of our approach it is now easy to consider the expected utility approach and mean-variance problems in continuous time in a common framework.

Furthermore, we like to point out that the formulation of the constrained problem (3.1) is very general. It fits for example pathwise constraints which could be expressed with the help of indicator functions. However, Theorem 3.1 is not applicable in this case, because assumption (3.3) is not satisfied. Therefore, the extension of Theorem 3.1 to such a case seems to be a desirable aspect for future research. This will also require a generalization of existing methods for unconstrained problems.

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