Optimal Portfolios with Defaultable Securities
A Firm Value Approach

Running title: Optimal Portfolios with Defaultable Securities

This version

April 24, 2002

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ABSTRACT: Credit risk is an important issue of current research in finance. While there is a lot of work on modelling credit risk and on valuing credit derivatives there is no work on continuous-time portfolio optimization with defautable securities. Therefore, in this paper we solve investment problems with defautable bonds and stocks. Besides, our approach can be applied to portfolio problems, where the investor has the opportunity to put her wealth into derivatives with counterparty risk or credit derivatives.

KEYWORDS: optimal portfolios, credit risk, elasticity, duration

JEL-CLASSIFICATION: G11

1 Introduction

In his pioneering work Merton (1969, 1971) considered an investor who allocates her wealth in stocks or a riskless money market account. However, the assumption is made that the interest rates are deterministic and that all assets are free of credit risk. Relaxing the first point was already addressed by Korn/Kraft (2001). The second point is rarely treated in literature.[1]

In this paper we will solve portfolio problems where the investor can put her money into defaultable assets such as corporate bonds. To model the credit risk we use a firm value approach which traces back to Black/Scholes (1973) and Merton (1974). This model which is called Merton's (firm value) model has the unpleasant feature that default can only occur at maturity. In contrast to that safety covenants give the bondholder the right to bankrupt, if the firm is doing poorly according to some standard. In firm value models this is modelled by a bankruptcy level which can be time dependent or even stochastic. The firm is forced to bankruptcy if the firm value falls to the bankruptcy level. This is done in Black/Cox (1976) for deterministic interest rates and in Briys/de Varenne (1997) for stochastic interest rates. Black/Cox (1976) also consider the case of subordinated bonds and restrictions on the financing of interest and dividend payments. Multiple further generalizations are addressed in Geske (1977), Mason/Bhattacharya (1981), Kim/Ramaswamy/Sundaresan (1993), Leland (1994), Longstaff/Schwartz (1995), Saa-Requejo/Santa-Clara (1999).

The main ingredient of Merton's model is the firm value. If the firm has not issued any bonds the firm value would coincide with the value of all stocks. Merton assumed that the firm additionally issued a zero bond. If the firm value at maturity is larger than the face value of the bond the stockholders will redeem the bond. Otherwise, the bond defaults and the firm passes over to the bondholders. Therefore the zero bond can be interpreted as a portfolio consisting of a cash position equal to the net present value of the face value and a short position in a put. Its exercise price is equal to the face value of the bond and the underlying is the firm value. Also, the share price is then given as a call on the firm value where again the strike equals the face value of the bond.

Consider a portfolio problem in which the investor can put her money into a stock, a (defaultable) bond and a riskless money market account. Actually, this means that the investor has the choice between two derivatives and the money market account. The main restriction in this problem results from the fact that the capital structure is explicitly modelled because the firm value equals the sum of the stock price and the bond value. Since the number of stock and bond is normalized to unity, the investor can buy at most one stock and one bond. In the ordinary formulation of portfolio problems such restrictions are not modelled and it is assumed that the demand of the investor is lower than the supply of the assets. In fact, there are also only a finite number of shares issued by a company although in theory no bound on the number of shares is considered.

The paper is organized as follows: In the next section we will first take up the simplifying approach of the usual problems and ignore the upper bound on the amount of stock and bond. This can be justified as a first order approximation for a small investor whose total wealth will (almost) never be enough to buy the total number of stocks and bonds. After having solved this problem we will look at the probability that the solution computed without the above constraints requires to hold at least one bond or stock of the company. Since the total number of stocks and bonds is normalized to unity, this means that the investor wants to buy at least the whole issue of stocks or bonds. In the third section, we solve the general constrained problem in the case of an investor who maximizes her terminal wealth with respect to a logarithmic utility function. In the fourth section some of the above mentioned generalizations of Merton's model are considered and the corresponding portfolio problems are solved. The paper concludes with a summary of our findings.

2 Optimal portfolios with defaultable bonds: The unconstrained case

As mentioned above it is a fact inherent in the Merton firm value model that the number of bonds and stocks of the issuing company is limited to one. Using the "small investor assumption" as a justification we will first ignore this fact and assume that there is no upper bound on the number of bonds and/or stocks.

We have seen that a defaultable bond can be interpreted as a portfolio consiting of a fixed payment F and a short position in a put with exercise price F. Therefore the value of the zero bond at maturity T_B is given by

$$B(T_B, v) = \min\{v, F\} = F - \max\{F - v, 0\}, \qquad v \geq 0. \tag{1}$$

The value of the stock equals the value of a call on the firm value with exercise price

F and maturity T_B, i.e.

$$S(T_B, v) = \max\{v - F, 0\}, \qquad v \geq 0. \tag{2}$$

Throughout the paper we make the assumption that the stock and the bond are traded continuously on a frictionless market. This does not mean that the investor may trade both assets, but it simplifies our presentations because then Black-Scholes-like formulae for the stock and the bond are valid. If one or both of these assets are not traded, a market price of risk comes into play. Although this would be an unpleasant feature for contingent claim pricing, it is a usual situation of portfolio optimization. This is the reason why our assumption is without loss of generality.

Optimizing a portfolio containing defaultable bonds can be considerably simplified if the *elasticity approach to portfolio optimization* by Kraft (2001) is used. Basically, this approach says that the optimal wealth process is determined by an optimal elasticity which is independent of a special asset. Therefore it consists of a kind of *two-step procedure*: First determine the optimal elasticity of the evolution of the investor's wealth process (for a given utility function) and then determine the portfolio process that achieves this elasticity (for a given tradable asset).

Hence, we start with a portfolio problem where the investor can put her wealth into the firm value and a money market account modelled by the SDE

$$dV(t) = V(t)\Big[\alpha dt + \sigma dW\Big], \qquad V(0) = v_0, \tag{3}$$

$$dM(t) = M(t) r dt, \qquad M(0) = 1. \tag{4}$$

The variable M denotes the value of the money market account and V the firm value which has a constant drift α and a constant volatility σ. Here and in the following W stands for a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is the usual P-augmentation of the natural filtration of W. In this section W is one dimensional. The variable r stands for the short rate which is held constant to simplify matters. Let $\lambda := \alpha - r$ be the excess return of the firm value.

Actually, the firm value is a non-traded asset. However, the elasticity approach tells us that we can solve the portfolio problem with the two investment opportunities V and M, and then compute how the optimal wealth process can be achieved by investing in the tradables, i.e. in the stock or bond issued by the company. As a consequence, we look at the following wealth equation[2]

$$dX(t) = X(t)\Big[(r + \pi_V \lambda) dt + \pi_V \sigma dW\Big], \tag{5}$$

$X(0) = x_0$, where X denotes the investor's total wealth and π_V stands for the percentage of the total wealth put into the firm value.

The classical examples for the portfolio problem

$$\max_{\pi} \mathrm{E}(u(X^{\pi}(T))) \tag{6}$$

are the choices of $u(x) = \ln(x)$ or $u(x) = \gamma^{-1}x^{\gamma}$. The variable T denotes the investment horizon. We assume that $T < T_B$ which means that during the investment period $[0, T]$ a default cannot occur. Of course, a low firm value indicates a high probability of default and a low bond price. In a latter section this assumption will be relaxed.

The optimal portfolio processes for (6) are well-known from the seminal papers by Merton (1969, 1971).

Proposition 2.1 (Merton's portfolio problem)
(i) For the logarithmic utility function $u(x) = \ln(x)$ the optimal portfolio process π_V^* for (6) is given as

$$\pi_V^*(t) = \frac{\lambda}{\sigma^2} \tag{7}$$

(ii) For the power utility function $u(x) = \gamma^{-1}x^{\gamma}$, $\gamma < 1$, $\gamma \neq 0$, the optimal portfolio process π_V^* for (6) is given as

$$\pi_V^*(t) = \frac{\lambda}{(1-\gamma)\cdot\sigma^2} \tag{8}$$

However, the firm value V shall not be tradable but claims on the firm value are. Although their prices are both non-linear functions of the firm value we can still use the results of the above Proposition to obtain the optimal wealth process. To demonstrate this idea assume that the investor can additionally invest her money in a contigent claim $C(t) = C(t, V(t))$ on the firm value. An application of Ito's rule and of the Black-Scholes partial differential equation results in the SDE

$$dC(t) = (rC + C_v V\lambda)dt + C_v V\sigma dW(t) \tag{9}$$

for the price of the claim. Here, C_v denotes the partial derivation of $C(t, v)$ with respect to the firm value. The wealth equation of this portfolio problem is given by

$$dX(t) = X\Big[(r + (\pi_V + \pi_C C^{-1}C_v V)\lambda)dt + (\pi_V + \pi_C C^{-1}C_v V)\sigma dW\Big] \tag{10}$$

where π_V denotes the percentage invested in the claim. This equation involves the elasticity of the claim with respect to the firm value which is defined by

$$\varepsilon_C = \frac{dC/C}{dV/V} := \frac{C_v V}{C}. \tag{11}$$

Note that for the corresponding elasticity of the firm value and the money market account we have $\varepsilon_V \equiv 1$ and $\varepsilon_M \equiv 0$, respectively. Therefore the term

$$\varepsilon := \pi_V + \pi_C C^{-1} C_v V = \pi_V \varepsilon_V + \pi_C \varepsilon_C \tag{12}$$

coincides with the static elasticity of the investor's portfolio.[3] Using this result the wealth equation simplifies to

$$dX = X\Big[(r + \varepsilon\lambda)dt + \varepsilon\sigma dW\Big]. \tag{13}$$

In this formulation the static portfolio elasticity ε is the control variable of the portfolio problem. Note that ε does not depend on a special asset. Since the wealth equations (5) and (13) only differ with respect to the notation of the control variable investment problems with contingent claims of the form $C(t) = C(t, V(t))$ can be solved as if the portfolio only contains the firm value and a cash position. This simple case is called a *reduced portfolio problem.* For example, if the investor maximizes her terminal utility at time T with respect to a logarithmic utility function $u(x) = \ln(x)$, $x > 0$, the above proposition yields that the optimal elasticity reads as

$$\varepsilon^*(t) = \frac{1}{1-\gamma}\frac{\lambda}{\sigma^2}. \tag{14}$$

Hence, any combination of firm value and claim which leads to the optimal elasticity ε^* can be choosen to achieve the optimal wealth process. This is the main result of the second step of the elasticity approach. Formally, we get that (π_V, π_C) has to be selected such that

$$\varepsilon^*(t) = \pi_V(t) + \pi_C(t) \cdot \varepsilon_C(t). \tag{15}$$

As in our case, the firm value V is not tradable, we must have $\pi_V \equiv 0$, hence

$$\varepsilon^*(t) = \pi_C(t) \cdot \varepsilon_C(t) \tag{16}$$

or

$$\pi_C^*(t) = \frac{\varepsilon^*(t)}{\varepsilon_C(t)}. \tag{17}$$

Since the stock and the bond in Merton's model are contingent claims on the firm value we obtain the following proposition:

Proposition 2.2 *Consider the portfolio problem (6) and Merton's firm value model.*

(i) If the investor is only allowed to invest into the money market account M and

the stocks S of the company then the optimal stock portfolio process π_S^* is given by

$$\pi_S^*(t) = \begin{cases} \frac{\lambda}{\sigma^2} \cdot \frac{S(t)}{S_v(t) \cdot V(t)} & = \quad \frac{\lambda}{\sigma^2} \cdot \frac{S(t)}{\mathcal{N}(d_1(t)) \cdot V(t)} \quad \text{if} \quad u(x) = \ln(x), \\ \frac{\lambda}{(1-\gamma)\sigma^2} \cdot \frac{S(t)}{S_v(t) \cdot V(t)} & = \quad \frac{\lambda}{(1-\gamma)\sigma^2} \cdot \frac{S(t)}{\mathcal{N}(d_1(t)) \cdot V(t)} \quad \text{if} \quad u(x) = \frac{1}{\gamma} x^\gamma, \end{cases} \tag{18}$$

where S denotes the stock price in the Merton's model, i.e

$$S(t) = V(t) \cdot \mathcal{N}(d_1(t)) - Fe^{-r(T-t)} \mathcal{N}(d_2(t)) \tag{19}$$

with

$$d_1(t) = \frac{\ln(\frac{V(t)}{F}) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{20}$$

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}, \tag{21}$$

and where $\mathcal{N}$ denotes the cumulative standard normal distribution function.

(ii) If the investor is only allowed to invest into the money market account M and the bonds B issued by the company then the optimal bond portfolio process π_B^* is given by

$$\pi_B^*(t) = \begin{cases} \frac{\lambda}{\sigma^2} \cdot \frac{B(t)}{B_v(t) \cdot V(t)} & = \quad \frac{\lambda}{\sigma^2} \cdot \frac{B(t)}{\mathcal{N}(-d_1(t)) \cdot V(t)} \quad \text{if} \quad u(x) = \ln(x), \\ \frac{\lambda}{(1-\gamma)\sigma^2} \cdot \frac{B(t)}{B_v(t) \cdot V(t)} & = \quad \frac{\lambda}{(1-\gamma)\sigma^2} \cdot \frac{B(t)}{\mathcal{N}(-d_1(t)) \cdot V(t)} \quad \text{if} \quad u(x) = \frac{1}{\gamma} x^\gamma, \end{cases} \tag{22}$$

where B denotes the bond price in Merton's model, i.e

$$B(t) = V(t) \cdot \mathcal{N}(-d_1(t)) + Fe^{-r(T-t)} \mathcal{N}(d_2(t)) \tag{23}$$

with d_1 and d_2 as in (i).

(iii) If the investor can put her wealth into the stock, the bond, and the money market account then every portfolio process (π_S, π_B) is optimal which matches the optimal elasticity, i.e.

$$\varepsilon^* = \pi_S \cdot \varepsilon_S + \pi_B \cdot \varepsilon_B. \tag{24}$$

Hence, the optimal strategy is not uniquely determined.

Proof. The proof is a direct consequence of Proposition 1, the form of the optimal elasticity ε^* in (17) and the price formulae for both stock and bond issued by the company. □

Remarks:

a) The above way of using elasticities generalizes ideas given in Korn/Trautmann (1998) to solve optimal portfolio problems with options.

b) We only choose the logarithmic and the power utility function for expositional convenience. Clearly, the results of the proposition are neither restricted to these utility functions nor to the maximation of terminal utility. The only ingredient that is needed is the optimal elasticity. As a consequence, the results hold for every utility function considered in Merton (1969,1971), Cox/Huang (1989, 1991) or Karatzas/Lehoczky/Shreve (1987).

c) By comparing the actual amount of money invested in the risky asset as computed in Proposition 1 and 2 we find

$$\pi_V^* \cdot X^* = \frac{\lambda}{\sigma^2} X^* > \frac{\lambda}{\sigma^2} \frac{B}{B_v V} X^* = \pi_B^* \cdot X^* \tag{25}$$

Thus, the optimal amount of money invested in the risky asset will always be lower if we trade in the bond of the company than if we would be able to trade the firm value. Therefore there is less money under default risk which seems to be a desirable feature. The same result holds for the stock S.

There remains to demonstrate that - given the investor's initial wealth is not too large - the original - but temporarily ingored - constraint on the number of stocks issued by the company can approximately be ignored. We first start with the problem when the investor can actually create a situation where she can "trade" the firm value by investing in both stock and bond and using the equation

$$V(t) = S(t, V(t)) + B(t, V(t)), \tag{26}$$

which we call accounting equation.

Proposition 2.3 (Relevance of the bound) *Let $x_0 > 0$ be the initial wealth of the investor who maximizes terminal wealth with respect to the power utility function $u(x) = \gamma^{-1}x^\gamma$. Further, assume that*

$$\frac{\lambda}{(1-\gamma)\sigma^2} \leq \frac{v_0}{x_0}, \tag{27}$$

i.e. at time $t = 0$ the investor does not want to buy more securities than issued. Then the probability that the optimal fraction of her wealth never exceeds the firm value V before T is given by

$$\mathcal{N}\Big(\frac{\ln(c) - aT}{|b|\sqrt{T}}\Big) - c^{\frac{2a}{b^2}} \mathcal{N}\Big(\frac{-\ln(c) - aT}{|b|\sqrt{T}}\Big) \tag{28}$$

with

$$a = -\lambda + 0.5\Big(\sigma^2 + \frac{\lambda^2}{(1-\gamma)\sigma^2}\Big), \quad b = \frac{\lambda}{(1-\gamma)\sigma} - \sigma, \quad c = \frac{\sigma^2 v_0 (1-\gamma)}{\lambda x_0}. \tag{29}$$

In the case of a logarithmic utility function $u(x) = \ln(x)$ the result is still valid with $\gamma = 0$.

Proof. As we have

$$V(t) = v_0 \cdot \exp\Big((r + \lambda - 0.5\sigma^2)t + \sigma W(t)\Big) \quad (30)$$

$$X^*(t) = x_0 \cdot \exp\Big((r + 0.5\tfrac{\lambda^2}{\sigma^2})t + \tfrac{\lambda}{\sigma} W(t)\Big) \quad (31)$$

the above probability is in the case of $u(x) = \ln(x)$ given by the probability

$$\begin{aligned}
&P\Big(\tfrac{\lambda}{\sigma^2} X^*(t) < V(t)\ \forall\, t \in [0,T]\Big)\\
&= P\Big(\max_{0\le t\le T} \tfrac{\lambda}{\sigma^2}\tfrac{x_0}{v_0} \exp\Big(0.5[\tfrac{\lambda^2}{\sigma^2} + \sigma^2 - 2\lambda]t + [\tfrac{\lambda}{\sigma} - \sigma]W(t)\Big) < 1\Big)\\
&= P\Big(\max_{0\le t\le T} \underbrace{[-\lambda + 0.5(\sigma^2 + \tfrac{\lambda^2}{\sigma^2})]}_{=a}t + \underbrace{[\tfrac{\lambda}{\sigma} - \sigma]}_{=b}W(t) < \ln(\underbrace{\tfrac{\sigma^2 v_0}{\lambda x_0}}_{=c})\Big)\\
&= P\Big(\max_{0\le t\le T} at + |b|W(t) < \ln(c)\Big)\\
&= P\Big(\max_{0\le t\le T} \tfrac{a}{|b|}t + W(t) < \tfrac{\ln(c)}{|b|}\Big)\\
&= \mathcal{N}\Big(\frac{\ln(c) - aT}{|b|\sqrt{T}}\Big) - c^{\frac{2a}{b^2}}\mathcal{N}\Big(\frac{-\ln(c) - aT}{|b|\sqrt{T}}\Big) \quad (32)
\end{aligned}$$

Note that bW and $|b|W$ have the same distribution. The last equation follows from Korn/Korn (2001, p. 168) and is thus a consequence of the reflection principle. The computations for the case of $u(x) = \gamma^{-1}x^\gamma$ are similar. □

Remarks:

a) It can be shown that for $x_0/v_0 \to 0$ which is the typical situation for a small investor the above probability approaches one quite fast. In table 1 there are some numerical values which illustrate this. We have used $\alpha = 0.15$, $r = 0.05$, $\sigma = 0.2$, and $\gamma = 0$. Thus, in the class relevant for a small investor, i.e. $x_0/v_0 \le 0.01$, the bound on available numbers of bond and stock is virtually irrelevant.

b) If however we think of big funds investing in the particular company then the above numbers also indicate that the constraints on the number of bonds and stocks cannot be ignored. We therefore consider such a constrained problem in the next section.

[Insert table 1]

The above proposition is only valid in the case where we can trade in both stocks and bonds of the issuing company. If we are only trading in either the stock or the bond then of course the probability increases that the constraints will be binding.

However, as both bond and stock prices are non-linear functions of the firm value, we cannot expect a closed form solution for the corresponding probabilities such as in the case considered in the proposition. To illustrate the behaviour of these probabilities we give some results obtained via Monte Carlo simulation. As above we have used $\alpha = 0.15$, $r = 0.05$, $\sigma = 0.2$, $\gamma = 0$, $v_0 = 1000$, and $F = 750$. Besides, it has been assumed that the maturity of the coporate securities lies one year after the investment horizon.

[Insert table 2]

Note that the capital structure of the firm - represented by the face falue F of the bond - becomes relevant, if the investor is only permitted to put her wealth in one of the corporate securities. Since only trading in the stock leads to similar results as before - in fact, every probability in the table 2 would be almost one - we omit the corresponding table. In contrast to this results a restriction to defaultable bonds heavily increases the probability of touching the barrier. The reason is as follows: Given an investor who only puts her wealth in corporate bonds or the money market account then her demand of bonds is only lower than the supply if[4]

$$\frac{1}{1-\gamma}\frac{\lambda}{B_v \sigma^2} \leq \frac{V}{X}. \tag{33}$$

Omitting discounting a face value of $F = 750$ and an initial firm value of $v_0 = 1000$ results in an equity ratio of 25%, which seems to be a realistic value. But in this case the put inherent in the bond is nearly worthless and the bond is nearly riskless. This leads to a delta of the bond which is almost zero. Hence, an investor, who wants to take an optimal risky position given by Merton's result, has to buy a large number of bonds. Therefore the upper bound on the number of bonds will be violated. This can be seen in inequality (33), where the delta of the bond stands in the denominator of the fraction on the left side. In contrast to that the delta of the stock is almost one, because the stock corresponds to a call, which is deeply in the money. Hence, investing in the stock leads to a substantially lower probability that the bound is touched.

We end this section with examples of defaultable assets where it suffices to consider the unconstrained case. Johnson/Stulz (1987) investigated assets with counterparty risk as for example vulnerable options. The model which they used is similar to Merton's model as they assume that the value of a vulnerable call at maturity is given by

$$\tilde{C}(T) = \min\{\tilde{V}(T), \max\{\tilde{S}(T) - K, 0\}\}, \tag{34}$$

where $\tilde{V}$ denotes the value of the assets of the call writer and $\tilde{S}$ the stock price. Note that the stock price here is not a contingent claim on a firm value but an originary asset as in Black/Scholes (1973) or Merton (1973) with the dynamics

$$d\tilde{S} = \tilde{S}\Big[\tilde{\alpha}dt + \tilde{\sigma}dW\Big], \quad \tilde{S}(0) = \tilde{s}_0. \tag{35}$$

Therefore one has to distinguish between the firm value V and the stock price S in Merton's model and the variable $\tilde{V}$ and the stock price $\tilde{S}$ in Johnson/Stulz (1987). Besides, it is important to note that in the present setting there is no link between the assets of the call writer and the number of calls written on the stock $\tilde{S}$. Therefore, the important but inconvenient feature of Merton's model that the number of bonds and stocks is bounded - this would correspond to bounds on the number of vulnerable calls - is not present in the model by Johnson/Stulz (1987). Consider an investor with a power utility function who can put her money into a money market account or a vulnerable call. Her optimal fraction invested in a vulnerable call is given by

$$\pi^*_{\tilde{C}}(t) = \frac{1}{1-\gamma}\frac{\tilde{\lambda}}{\tilde{\sigma}^2}\frac{1}{\varepsilon_{\tilde{C}}(t)}. \tag{36}$$

In general, there exists no closed-form solution for the price $\tilde{C}$ of a vulnerable call. Nevertheless, our result stays valid, but one has to compute the elasticity $\varepsilon_{\tilde{C}}$ numerically. For the special case of covered call writing, which means that $\tilde{V}(t) = \rho \cdot \tilde{S}(t)$, $0 < \rho < 1$, Johnson/Stulz (1987) gave a closed-form solution which reads as[5]

$$\tilde{C}(0) = C(\tilde{S}(0), K) - (1-\rho) \cdot C(\tilde{S}(0), K/(1-\rho)), \tag{37}$$

where $C(s, K)$ denotes the ordinary Black-Scholes price of a call with strike price K given that the stock price equals s. Hence, we have

$$\frac{\partial \tilde{C}(t)}{\partial \tilde{S}} = \mathcal{N}(d_1'(t)) - (1-\rho) \cdot \mathcal{N}(d_1''(t)) \tag{38}$$

with

$$d_1'(t) = \frac{\ln(\frac{\tilde{S}(t)}{K}) + (r + 0.5\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}}, \quad d_1''(t) = \frac{\ln(\frac{(1-\rho)\tilde{S}(t)}{K}) + (r + 0.5\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}}. \tag{39}$$

Using this result the optimal fraction invested in the vulnerable option has the following form

$$\pi^*_{\tilde{C}}(t) = \frac{1}{1-\gamma}\frac{\tilde{\lambda}}{\tilde{\sigma}^2}\frac{\tilde{S}(t)}{\tilde{C}(t) \cdot (\mathcal{N}(d_1'(t)) - (1-\rho) \cdot \mathcal{N}(d_1''(t)))}. \tag{40}$$

Note that the optimal fraction is positive. Clearly, analoguous results can be calculated in the model by Hull/White (1995), who were able to prove a Black-Scholes-like formula for vulnerable options in a special case of their model.

Another important example of defaultable claims for which no restrictions has to be considered are credit derivatives.

3 Optimal Portfolios with defaultable Bonds: The Constrained Case

As already announced, we now consider Merton's firm value model including all the constraints on the number of bonds and shares issued by the company. Denoting the number of stocks and bonds in the investor's portfolio with φ_S and φ_B, respectively, this will lead to the restrictions

$$|\varphi_S(t)| \leq 1 \qquad \text{and} \qquad |\varphi_B(t)| \leq 1. \tag{41}$$

If short sales are prohibited these restrictions reads as

$$0 \leq \varphi_S(t) \leq 1 \qquad \text{and} \qquad 0 \leq \varphi_B(t) \leq 1. \tag{42}$$

Note that we have normalized the number of shares and bonds issued by the company to unity. Since

$$\varphi_S = \frac{\pi_S \cdot X}{S} \qquad \text{and} \qquad \varphi_B = \frac{\pi_B \cdot X}{B}, \tag{43}$$

where X denotes the total wealth of the investor, we get

$$|\pi_S| \leq \frac{S}{X} \qquad \text{and} \qquad |\pi_B| \leq \frac{B}{X} \tag{44}$$

or

$$0 \leq \pi_S \leq \frac{S}{X} \qquad \text{and} \qquad 0 \leq \pi_B \leq \frac{B}{X}, \tag{45}$$

respectively. Hence, the portfolio process (π_S, π_B) is bounded by a stochastic process which itself depends on the control. Such boundaries even do not fall in the classes treated by Karatzas/Cvitanic (1992). Obviously, these boundaries will be restrictive if the investor's wealth is large enough. Without loss of generality we concentrate on the case when short sales are prohibited.

As before we consider a portfolio problem in which the investor can put her money into a riskless money market account M, a stock S, and/or a (defaultable) bond

B. However, we restrict to the case of an investor who maximizes her terminal wealth up to time T, $T < T_B$, with respect to a logarithmic utility function. To solve the problem we use the elasticity approach. Hence, the wealth equation has the following form

$$dX(t) = X(t)\Big[(r + \varepsilon(t)\lambda)dt + \varepsilon(t)\sigma dW(t)\Big]. \tag{46}$$

Recall that the elasticities of the stock and the bond are

$$\varepsilon_S = \frac{\partial S}{\partial V}\frac{V}{S} = \mathcal{N}(d_1)\cdot\frac{V}{S}, \qquad \varepsilon_B = \frac{\partial B}{\partial V}\frac{V}{V} = \mathcal{N}(-d_1)\cdot\frac{V}{B}. \tag{47}$$

If the investor can trade in the stock and in the bond, at time $t \in [0, T]$ she can attain any elasticities $\varepsilon(t)$ with

$$\varepsilon(t) = \pi_S(t)\cdot\varepsilon_S(t) + \pi_B(t)\cdot\varepsilon_B(t) \tag{48}$$

where the restrictions (45) concerning the portfolio process $\pi = (\pi_S, \pi_B)$ have to be observed. Plugging (43) in (48) leads to

$$\varepsilon(t) = \tfrac{V(t)}{X(t)}\Big(\varphi_S(t)S_v(t) + \varphi_B(t)B_v(t)\Big) \tag{49}$$

As S_v, $B_v > 0$ the investor can achieve elasticites $\varepsilon(t)$ which belong to the following intervall

$$[0, \tfrac{V(t)}{X(t)}(S_v + B_v)] = [0, \tfrac{V(t)}{X(t)}]. \tag{50}$$

Note that the equality results from the accounting equation $S + B = V$. Hence, we have to solve the following optimization problem, where $\mathcal{A}(0, x_0)$ denotes the set of all admissible controls[6] given the initial condition $(0, x_0)$

$$\max_{\varepsilon(\cdot)\in\mathcal{A}^*_B(0,x_0)} \mathrm{E}(\ln X^\varepsilon(T)) \tag{51}$$

with

$$dX^\varepsilon(t) = X^\varepsilon(t)\Big[(r + \varepsilon(t)\lambda)dt + \varepsilon(t)\sigma dW(t)\Big], \tag{52}$$

$$X^\varepsilon(0) = x_0 \tag{53}$$

and

$$\mathcal{A}^*_B(0, x_0) := \Big\{\varepsilon(\cdot) \in \mathcal{A}(0, x_0)\ :\ X^\varepsilon(t) \geq 0 \text{ and } \varepsilon(t) \in [0, \mathcal{U}(t)] \text{ for } t \in [0, T]\Big\}, \tag{54}$$

where $\mathcal{U}$ denotes an adapted upper bound for the attainable elasticities. As shown above, we have $\mathcal{U}(t) = V(t)/X^\varepsilon(t)$, if the investor can trade in both stocks and bonds. To point out the dependency of X on ε we have writen X^ε. In the following, as before, we will mostly omit the superindex. It is important to mention that the special form of the coefficients in the wealth equation will guarantee the positivity of $X^{\varepsilon}(t)$.[7] Besides, the reader should be aware of the fact that in the present setting the unconstrained optimal elasticity is not always attainable. To emphasize this we call the elasticity which solves the optimisation problem (51) the optimal *attainable* elasticity.

The following proposition gives the solution to the problem (51). Hence, the first step of the elasticity approach is taken.

Proposition 3.1 (Optimal Elasticity in the Merton Model) *Consider the portfolio problem (51) and Merton's firm value model. Then the optimal attainable elasticity reads as follows*

$$\varepsilon^*(t) = \begin{cases} \frac{\lambda}{\sigma^2}, & \textit{if} \quad \mathcal{U}(t) \geq \frac{\lambda}{\sigma^2}, \\ \mathcal{U}(t), & \textit{if} \quad \mathcal{U}(t) < \frac{\lambda}{\sigma^2}. \end{cases} \tag{55}$$

Proof. The solution to the wealth equation (46) is given by

$$X(t) = x_0 \cdot \exp\Big(\int_0^t r + \varepsilon(s)\lambda - 0.5\varepsilon^2(s)\sigma^2 \, ds + \int_0^t \varepsilon(s)\sigma \, dW(s) \Big). \tag{56}$$

Hence, the investor's expected terminal utility reads as

$$\mathrm{E}(\ln X(T)) = \ln x_0 + rT + \mathrm{E}\Big(\int_0^t \varepsilon(s)\lambda - 0.5\varepsilon^2(s)\sigma^2 \, ds \Big). \tag{57}$$

Note that due to the boundedness of $\varepsilon(\cdot)$ the expectation of the Ito integral vanishes. Completing the square in the above integrand leads to

$$\begin{aligned} \mathrm{E}(\ln X(T)) &= \ln x_0 + rT + 0.5\tfrac{\lambda^2}{\sigma^2}T - 0.5 \cdot \mathrm{E}\Big(\int_0^t \big[\varepsilon(s)\sigma - \tfrac{\lambda}{\sigma}\big]^2 ds \Big) & (58) \\ &= \ln x_0 + rT + 0.5\tfrac{\lambda^2}{\sigma^2}T - 0.5\sigma^2 ||\varepsilon - \tfrac{\lambda}{\sigma^2}||^2, & (59) \end{aligned}$$

where $||\cdot||$ denotes the norm of the space $\mathcal{L}^2$. Therefore, the utility reaches a maximum when the norm is minimal. Using an orthogonal projection argument the norm becomes minimal if

$$|\varepsilon(t) - \tfrac{\lambda}{\sigma^2}| \tag{60}$$

reaches its minimum at each time instant $t \in [0, T]$. Since $\varepsilon(t) \in [0, \mathcal{U}(t)]$, this leads to (55). Note in particular that $X(t)$ does not dependent on $\varepsilon(t)$. □

Remark:

a) We want to point out that the orthogonal projection argument in the above proof does not work if we assume that the preferences of the investor are governed by a power utility function as then the corresponding integrand is not independent of the wealth process X itself.

b) The above result does not only seem to be intuitive, it can also be applied to very general upper bounds which can even be dependent on the control ε itself.

By specifying the upper bound $\mathcal{U}$ the optimal portfolio processes with defaultable securities can be computed. This corresponds to the second step of the elasticity approach. The respective results are summerized in the following corollary.

Corollary 3.1 (Optimal Portfolios in the Merton Model)
(i) If the investor may put her money into the riskless money market account and the stock then we have $\mathcal{U}(t) = V(t) \cdot S_v(t)/X(t)$. *The optimal fraction invested in the stock is uniquely determined and equal to*

$$\pi_S^*(t) = \frac{\varepsilon^*(t)}{\varepsilon_S(t)} = \begin{cases} \frac{\lambda}{\sigma^2}\frac{S(t)}{V(t)S_v(t)}, & \text{if} \quad \frac{V(t)}{X(t)}S_v(t) \geq \frac{\lambda}{\sigma^2}, \\ \frac{S(t)}{X(t)}, & \text{if} \quad \frac{V(t)}{X(t)}S_v(t) < \frac{\lambda}{\sigma^2}, \end{cases} \tag{61}$$

where $S_v = N(d_1)$.

(ii) If the investor may put her money into the riskless money market account and the defaultable bond then the we have $\mathcal{U}(t) = V(t) \cdot B_v(t)/X(t)$. *The optimal fraction invested in the bond is uniquely determined and equal to*

$$\pi_B^*(t) = \frac{\varepsilon^*(t)}{\varepsilon_B(t)} = \begin{cases} \frac{\lambda}{\sigma^2}\frac{B(t)}{V(t)B_v(t)}, & \text{if} \quad \frac{V(t)}{X(t)}B_v(t) \geq \frac{\lambda}{\sigma^2}, \\ \frac{B(t)}{X(t)}, & \text{if} \quad \frac{V(t)}{X(t)}B_v(t) < \frac{\lambda}{\sigma^2}, \end{cases} \tag{62}$$

where $B_v = N(-d_1)$.

(iii) If the investor is allowed to split up her money in the riskless money market account, the defaultable bond, and the stock then $\mathcal{U}(t) = V(t)/X(t)$. *The optimal portfolio process is not uniquely determined, but every combination of stock and bond is optimal which leads to optimal elasticity.*

Proof. (i) Since the investor can only put her money into the money market account and the stock, the attainable elasticities are given by

$$\varepsilon(t) = \frac{V(t)}{X(t)}\varphi_S(t)S_v(t), \quad 0 \leq \varphi_S \leq 1. \tag{63}$$

Therefore we get $\mathcal{U}(t) = V(t) \cdot S_v(t)/X(t)$. Inserting the definition of φ_S in (63) leads to (61).

As equation (63) can uniquely be solved for φ_S the equation (61) is proved.

(ii) The proof of (ii) is similar to the proof of (ii).

(iii) The upper bound follows from (50). Since there is one degree of freedom in solving the equation (49), the portfolio process is not uniquely determined. □

Remark: The situation described in (iii) has a nice geometric interpretation. The optimal combinations of stock and bond lie on the straight line $\mathcal{G}$ given by

$$\pi_S = \frac{1}{\varepsilon_S}\left(\frac{\lambda}{\sigma^2} - \pi_B \varepsilon_B\right), \tag{64}$$

whereas the range of an attainable strategy (π_S, π_B) corresponds to the rectangle

$$\mathcal{R} := [0, \tfrac{S}{X}] \times [0, \tfrac{B}{X}]. \tag{65}$$

Hence, the unconstrained optimal elasticity λ/σ^2 is attainable, if the straight line and the rectangle have at least one point in common, i.e. $\mathcal{G} \cap \mathcal{R} \neq \emptyset$.

As mentioned in the last section, the ratio v_0/x_0 between the initial firm value and the initial wealth of the investor is crucial for the relevance of the restriction. An investor, who has a lot of funds to invest - i.e. her initial wealth is large -, will not be able to buy a position which perfectly tracks the unconstrained optimal elasticity. Hence, a *tracking error* occurs. Clearly, this problem will be less servere if the firm value of the company becomes larger. Nevertheless, the restricions cannot be neglected in the case of well funded investment funds. Although we have assumed that all investors are price takers it is reasonable to conclude that such a situation will lead to increasing prices of the traded firm securities. This can be seen as one explanation of the bubble which has arised e.g. at the American NASDAQ or the German Neuer Markt. Obviously, in these market segments default risk is an important issue which has to be taken into account. But it does not seem to be a tenable assumption that default can only occur after the investment horizon. We address this point in the next section.

4 Generalizations of Merton's model

In this section we make the same main assumptions as in the last one. Especially, we consider an investor with a logarithmic utility function. Besides, it is assumed for convenience that risk-neutral valuation formulae for the securities issued by the firm are valid. However, we look at some generalizations of Merton's model.

4.1 Black-Cox Model

In contrast to Black/Scholes (1973) and Merton (1974), Black/Cox (1976) consider the impact of safety covenants on the value of the firm's securities. This contractual provision gives the bondholder the right to trigger default when the firm value touches a lower bound

$$L(t) = k \cdot e^{-\kappa(T_B - t)} \tag{66}$$

with constants $k, \kappa > 0$. Let $\tau := \inf\{t \geq 0 : V(t) = L(t)\}$ be the corresponding stopping time. If the default event occurs during the life of the bond the bondholders immediately obtain the ownership of the firm's whole assets. Otherwise, the terminal value of the bond is identical to the value in Merton's model. Hence, the defaultable bond corresponds to a portfolio of barrier derivatives with the curved boundary L and a maturity equal to the maturity T_B of the bond. More precisely, the corporate bond has the following value at T_B

$$B(T_B, V(T_B)) = F \cdot P_{DO}(T_B) - \max\{F - V(T_B), 0\}_{DO} + H(T_B), \tag{67}$$

where P_{DO} denotes a down-and-out bond with knock-out barrier L and H a cash-at-hit option which will pay the firm value if the barrier is touched, i.e. $H(\tau) = k \cdot e^{-\kappa(T_B - \tau)}$ given that $\tau < T_B$. The second term stands for a down-and-out put. If the down-and-out properties are neglected, the first two terms are equal to the final bond value in Merton's model. However, the last term is new. The reader should be aware of the difference between a down-and-in bond and cash-at-hit option. Whereas the first leads to a constant payment *at maturity*, if a barrier is touched during the life of down-and-in bond, the second *immediately* pays a constant ammount which leads to the final value

$$H(T_B) = \begin{cases} k \cdot e^{(r-\kappa)\cdot(T_B - \tau)} & if \quad \tau < T_B, \\ 0 & if \quad \tau > T_B. \end{cases} \tag{68}$$

As a consequence, the cash-at-hit option only corresponds to a down-and-in bond, if $r = \kappa$, i.e. if the growth rate of the boundary equals the compounding rate of the riskless money market account. In this special case the stochastic payment date of the cash-at-hit option is irrelevant. We call $L(t) = ke^{-r(T_B - t)}$ a *discounted barrier*.

For $k \leq F$ the value of the bond at time $t \in [0, \min\{T_B, \tau\}]$ is given by[8]

$$\begin{aligned} B(t) &= Fe^{-r(T_B - t)}\Big[\mathcal{N}(z_1(t)) - y^{2\theta - 2}(t)\mathcal{N}(z_2(t))\Big] \\ &\quad + V(t)\Big[\mathcal{N}(-z_3(t)) + y^{2\theta}(t)\mathcal{N}(z_4(t))\Big], \end{aligned} \tag{69}$$

where

$$z_{1/3}(t) = \frac{\ln(\frac{V(t)}{F}) + (r \mp 0.5\sigma^2)(T_B - t)}{\sigma\sqrt{T_B - t}}, \tag{70}$$

$$z_{2/4}(t) = \frac{\ln(\frac{V(t)}{F}) + 2\ln(y(t)) + (r \mp 0.5\sigma^2)(T_B - t)}{\sigma\sqrt{T_B - t}}, \tag{71}$$

$y(t) = ke^{-\kappa(T_B - t)}/V(t)$, and $\theta = (r - \kappa + 0.5\sigma^2)/\sigma^2$. As in the former sections, the elasticity of the bond plays an important role for the optimal portfolio process. Therefore we have to compute the derivative of the bond value with respect to the firm value. Rearranging of (69) leads to a representation which proves very useful for these matters:

$$B(t) = \underbrace{Fe^{-r(T_B - t)} - \Big[Fe^{-r(T_B - t)}\mathcal{N}(-z_1(t)) - V(t)\mathcal{N}(-z_3(t))\Big]}_{\text{Merton's bond price}} \tag{72}$$

$$+y^{2\theta - 2}(t)\Big[\underbrace{V(t)y^2(t)\mathcal{N}(z_4(t)) - Fe^{-r(T_B - t)}\mathcal{N}(z_2(t))}_{=C(t)}\Big]. \tag{73}$$

The bond price equals the sum of the bond price in Merton's model, which itself is the difference between a put price and the price of a riskless bond, and a correction term, which is equal to a number of calls with the fictitious underlying $V(t)y^2(t)$. The corresponding call value is denoted by C.

Hence, the derivative of the bond with respect to the firm value is given by

$$B_v(t) = \mathcal{N}(-z_3(t)) - y^{2\theta}(t)\Big[\tfrac{2\theta - 2}{V(t)y^2(t)}C(t) + \mathcal{N}(z_4(t))\Big]. \tag{74}$$

The first term corresponds to the derivative in Merton's model the second term is a correction due to the safety covenant. It can be shown that this derivative is positive and smaller than one.

If the lower bound is not reached, the value of the stocks at maturity T_B of the corporate bond is equal to the value in Merton's model, otherwise the value equals zero. Hence, we get

$$S(t, V(t)) = \max\{V(t) - F, 0\}_{DO}, \tag{75}$$

i.e. the stock price equals the price of a down-and-out call. As we have that $0 \leq B_v \leq 1$, the accounting equation leads to $0 \leq S_v \leq 1$.

As in Merton's model the bond price corresponds to a portfolio of derivatives. Equivalentely, the stock is modelled as a single derivative. Hence, an investor will try to track an optimal (attainable) elasticity. However, if the firm is triggered to default before the investment horizon T, the firm assets are handed over to the bondholders. Given that they only want to invest in bonds they are forced to invest the proceeds in the money market account. Therefore, the only attainable elasticity after default is equal to zero. Keeping this in mind the portfolio problem of an investor reads as

$$\max_{\varepsilon(\cdot)\in\mathcal{A}'_B(0,x_0)} \mathrm{E}(\ln X^{\varepsilon}(T))\} \tag{76}$$

with

$$\begin{aligned} dX^{\varepsilon}(t) &= X^{\varepsilon}(t)\Big[(r+\varepsilon(t)\lambda)dt + \varepsilon(t)\sigma dW(t)\Big], & (77)\\ X^{\varepsilon}(0) &= x_0 & (78)\end{aligned}$$

and

$$\mathcal{A}'_B(0,x_0) := \Big\{\varepsilon(\cdot)\in\mathcal{A}^*(0,x_0)\,:\,\varepsilon(t)=0\,\forall\,t\in[\tau,T]\Big\}. \tag{79}$$

Note that the elements of $\mathcal{A}'_B(0,x_0)$ are processes which are killed at the stopping time τ. Hence, $\mathcal{A}'_B(0,x_0)\subset\mathcal{A}^*_B(0,x_0)$, which implies that all elements of $\mathcal{A}'_B(0,x_0)$ are admissible controls. We want to emphasize that the partial derivatives of both stock and bond with respect to the firm value are positive. Hence, only positive elasticities are attainable. The solution to this problem is given in the following proposition:

Proposition 4.1 (Optimal elasticity in the Black-Cox Model)
Consider the portfolio problem (76) and the Black-Cox model. Then the optimal attainable elasticity is equal to

$$\varepsilon^*(t) = \begin{cases} \frac{\lambda}{\sigma^2}, & \text{if} \quad \tau > t \text{ and } \mathcal{U}(t)\geq\frac{\lambda}{\sigma^2},\\ \mathcal{U}(t), & \text{if} \quad \tau > t \text{ and } \mathcal{U}(t)<\frac{\lambda}{\sigma^2},\\ 0, & \text{if} \quad \tau\leq t.\end{cases} \tag{80}$$

Proof. Similar arguments as in the proof of proposition 3.1 apply. □

Since the investor has a logarithmic utility function we can apply a pointwise maximation argument. Hence, for the optimality of an elasticity up to a possible default it is irrelevant, if/when a default occurs. As a consequence, the portfolio stays optimal regardless, whether the firm is liquidated after default or it is taken over or it is reorganized and new tradable stocks are issued etc. This is the reason, why it suffices to consider the optimization problem (76), where the attainable elasticity ε is killed after default. Without this independence one has to take into account, whether there are any firm securities after default which the investor can trade.

Using this result of proposition 4.1 we can solve for the optimal portfolio fractions:

Corollary 4.1 (Optimal portfolios in the Black-Cox Model)

(i) If the investor may put her money into the riskless money market account and the stock then we have $\mathcal{U}(t) = V(t) \cdot S_v(t)/X(t)$*,* $0 \le t < \tau$*. The optimal fraction invested in the stock is uniquely determined and equal to*

$$\pi_S^*(t) = \frac{\varepsilon^*(t)}{\varepsilon_S(t)} = \begin{cases} \frac{\lambda}{\sigma^2}\frac{S(t)}{V(t)S_v(t)}, & \text{if} \quad \tau > t \text{ and } \frac{V(t)}{X(t)}S_v(t) \ge \frac{\lambda}{\sigma^2}, \\ \frac{S(t)}{X(t)}, & \text{if} \quad \tau > t \text{ and } \frac{V(t)}{X(t)}S_v(t) < \frac{\lambda}{\sigma^2}, \\ 0, & \text{if} \quad \tau \le t, \end{cases} \tag{81}$$

where $S_v(t) = \mathcal{N}(z_3(t)) + y^{2\theta}(t)\left[\frac{2\theta-2}{V(t)y^2(t)}C(t) + \mathcal{N}(z_4(t))\right]$.

(ii) If the investor may put her money into the riskless money market account and the defaultable bond then the we have $\mathcal{U}(t) = V(t) \cdot B_v(t)/X(t)$*,* $0 \le t < \tau$*. The optimal fraction invested in the bond is uniquely determined and equal to*

$$\pi_B^*(t) = \frac{\varepsilon^*(t)}{\varepsilon_B(t)} = \begin{cases} \frac{\lambda}{\sigma^2}\frac{B(t)}{V(t)B_v(t)}, & \text{if} \quad \tau > t \text{ and } \frac{V(t)}{X(t)}B_v(t) \ge \frac{\lambda}{\sigma^2}, \\ \frac{B(t)}{X(t)}, & \text{if} \quad \tau > t \text{ and } \frac{V(t)}{X(t)}B_v(t) < \frac{\lambda}{\sigma^2}, \\ 0, & \text{if} \quad \tau \le t, \end{cases} \tag{82}$$

where $B_v(t) = \mathcal{N}(-z_3(t)) - y^{2\theta}(t)\left[\frac{2\theta-2}{V(t)y^2(t)}C(t) + \mathcal{N}(z_4(t))\right]$.

(iii) If the investor is allowed to split up her money in the riskless money market account, the defaultable bond, and the stock then $\mathcal{U}(t) = V(t)/X(t)$*,* $0 \le t < \tau$*. The optimal portfolio process is not uniquely determined. Every combination of stock and bond is optimal which leads to optimal elasticity.*

Remark: To motivate the elasticity approach we assumed the validity of the Black-Scholes partial differential equation. Note that barrier derivatives meet this equation up to the stopping time τ. This is the reason why the elasticity approach is still applicable in the Black-Cox model.

As mentioned in the introduction Black/Cox (1976) also considered the case of subordinated bonds (syn. junior bonds). Since the value of such corporate bonds can be expressed as the difference between the value of two senior bonds, a junior bond also corresponds to a contingent claim on the firm value. Therefore a portfolio problem with junior bonds can be treated as before.

4.2 Generalized Briys-de Varenne Model

One shortcoming of the Black-Cox model results from their assumption that the interest rates are deterministic although corporate bonds are significantly influenced by interest rate risk.[9] To overcome this drawback Briys/de Varenne (1997) modelled a stochastic short rate using an extended Vasicek model. Clearly, this restriction is made to get closed-form solutions for the values of the firm securities. Since we do not need this assumption, we simply assume that the dynamics of the short rate are governed by

$$dr(t) = a(t)dt + b(t)dW_r(t), \quad r(0) = 0, \tag{83}$$

where with a slight abuse of notation the drift $a(t) = a(t, r(t))$ and the volatility $b(t) = b(t, r(t))$ are measurable and sufficiently integrable, so that the SDE (83) has a unique solution.[10] Hence, the models by Vasicek (1977), Dothan (1978), Cox/Ingersoll/Ross (1985), Ho/Lee (1986), and Black/Derman/Toy (1990) are governed as special cases. Additionally, let $b \neq 0$ almost surely.

The dynamics of the firm value are modelled by

$$dV(t) = V(t)\Big[\big(r(t) + \lambda_V(t)\big)dt + \sigma_V(t)dW_V(t) + \sigma_r(t)dW_r(t)\Big], \tag{84}$$

where the processes $\lambda_V(t) = \lambda_V(t, V(t), r(t))$, $\sigma_V(t) = \sigma_V(t, V(t), r(t))$, and $\sigma_r(t) = \sigma_r(t, V(t), r(t))$ are measurable and sufficiently integrable, so that the SDE has a unique solution. Besides, let σ_V and σ_r be bounded away from zero.

As Black/Cox (1976), Briys/de Varenne (1997) allow for immediate default if a lower (discounted) boundary

$$L(t) = k \cdot F \cdot P(t, T_B) \tag{85}$$

is reached, where $P(t, T_B)$ denotes the price of a riskless bond with maturity T_B at time t. If there does not occur a default prior to time T_B, it is assumed that the corporate bond has the same value as in the Merton or the Black-Cox model. Note that for deterministic interest rates the above boundary is equal to $kFe^{-r(T_B - t)}$ which corresponds to a special case of the default boundary in the Black-Cox model. Hence, the value of the corporate bond can now be expressed as the value of a portfolio consisting of a down-and-out put, a down-and-out bond, and a down-and-in bond. More precisely, the value of the defaultable bond at time T_B is given by[11]

$$B(T_B, V(T_B)) = F \cdot P_{DO}(T_B) - \max\{F - V(T_B), 0\}_{DO} + k \cdot F \cdot P_{DI}(T_B), \tag{86}$$

where P_{DI} denotes a down-and-in bond with knock-in barrier L. Recall that in the case of a discounted barrier the value of a cash-at-hit option and a down-and-in bond coincide. Besides, we have $P_{DO}(T_B) = \mathcal{I}(\tau > T_B)$ and $P_{DI}(T_B) = \mathcal{I}(\tau \leq T_B)$, where $\mathcal{I}(E)$ denotes the indicator function for E and $\tau = \inf\{t \geq 0 : V(t) = L(t)\}$. The stock price at maturity T_B of the corporate bond reads as

$$S(T_B, V(T_B)) = \max\{V(T_B) - F, 0\}_{DO}, \tag{87}$$

i.e. the stock is identical to a down-and-out call with a discounted barrier. Since both stock and bond are contingent claims depending on the firm value V and the short rate r, we proceed by extending the elasticity approach to stochastic interest rates. In the last sections we saw that the elasticity of the contingent claims with respect to firm value plays an important role. Now, a second Brownian motion has entered the stage, namely the one which governs the interest rate risk. As a consequence, the sensitivity of a claim with respect to interest rate risk should become relevant. This sensitivity is known as the duration of the claim and is defined by $D_C = C_r/C$.[12]

Keeping this in mind we consider a contingent claim $C(t) = C(t, V(t), r(t))$. Applying Ito's lemma we get

$$\begin{aligned} dC &= C_t dt + C_v dV + C_r dr + 0.5 C_{vv} d < V > + 0.5 C_{rr} d < r > + C_{vr} d < V, r > \\ &= \Big(C_t + C_v V(r + \lambda_V) + a C_r + 0.5 C_{vv} V^2 (\sigma_V^2 + \sigma_r^2) + 0.5 C_{rr} b^2 \\ &\quad + b\sigma_r V C_{vr}\Big) dt + C_v V \sigma_V dW_V + (C_v V \sigma_r + C_r b) dW_r. \end{aligned} \tag{88}$$

Besides, the claim price satisfies a generalized Black-Scholes partial differential equation[13]

$$\begin{aligned} &C_t + \Big(r + \lambda_V - \sigma_V \zeta_V - \sigma_r \zeta_r\Big) v C_v + \Big(a - \zeta_r b\Big) C_r \\ &+ 0.5\Big((\sigma_V^2 + \sigma_r^2) v^2 C_{vv} + 2b\sigma_r v C_{vr} + b^2 C_{rr}\Big) - rC \quad = \quad 0, \end{aligned} \tag{89}$$

where $\zeta_V := \lambda_V / \sigma_V$ denotes the market price of risk of the firm value and $\zeta_r(t) = \zeta_r(t, r(t))$ stands for the market price of risk of the market for riskless bonds. We assume that the latter is measurable and integrable. The market price ζ_r stays as long unspecified as one does not assume that an interest rate sensitive claim, e.g. a riskless bond, is traded. Defining the aggregate excess return $\lambda := \sigma_V \zeta_V + \sigma_r \zeta_r$ and plugging the partial differential equation in (88) gives

$$dC = C\Big[\Big(r + \lambda \varepsilon_C + b\zeta_r D_C\Big) dt + \sigma_V \varepsilon_C dW_V + \Big(\sigma_r \varepsilon_C + b D_C\Big) dW_r\Big], \tag{90}$$

where $\varepsilon_C := C_v V/C$ denotes the elasticity of the claim and $D_C := C_r/C$ the duration of the claim. Let π_C denote the fraction invested in the contingent claim, then the number of contingent claims is given by $\varphi_C = \pi_C \cdot C/X$. If an investor can only split up her wealth in a riskless money market account and the claim, the amount of money invested in the account can be computed as $\varphi_M = (1-\pi_C)\cdot M/X$. Therefore, we arrive at the wealth equation

$$\begin{aligned}
dX &= \varphi_M dM + \varphi_C dC \\
&= (1-\pi_C)Xrdt \\
&\quad +\pi_C X\Big[\big(r+\lambda\varepsilon_C + b\zeta_r D_C\big)dt + \sigma_V\varepsilon_C dW_V + \big(\sigma_r\varepsilon_C + bD_C\big)dW_r\Big], \\
&= X\Big[\big(r+\lambda\varepsilon + b\zeta_r D\big)dt + \sigma_V\varepsilon dW_V + \big(\sigma_r\varepsilon + bD\big)dW_r,\Big] \qquad (91)
\end{aligned}$$

where ε and D denote the static elasticity and duration of the portfolio, i.e.

$$\varepsilon = \pi_C\cdot\varepsilon_C + (1-\pi_C)\cdot\varepsilon_M \quad \text{and} \quad D = \pi_C\cdot D_C + (1-\pi_C)\cdot D_M. \qquad (92)$$

Recall that $\varepsilon_M = D_M = 0$. This procedure can be generalized to an arbitrary number of claims $C(t) = C(t, V(t), r(t))$. Hence, the elasticity approach applies in the case of stochastic interest rates. The relevant controls are the elasticity and the duration.

Given an investor with a logarithmic utility function the optimal portfolio problem reads as

$$\max_{(\varepsilon(\cdot),D(\cdot))\in\mathcal{A}'_B(0,x_0)} \mathrm{E}(\ln X^{\varepsilon,D}(T))\} \qquad (93)$$

with

$$\begin{aligned}
dX^{\varepsilon,D} &= X\Big[\Big(r + (\sigma_V\zeta_V + \sigma_r\zeta_r)\varepsilon + \zeta_r bD\Big)dt + \sigma_V\varepsilon dW_V + (\sigma_r\varepsilon + bD)dW_r\Big], \\
X^{\varepsilon,D}(0) &= x_0 \qquad (94)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}'_B(0,x_0) &:= \Big\{(\varepsilon(\cdot),D(\cdot))\in\mathcal{A}(0,x_0) \,:\, X^\varepsilon(t)\geq 0,\ \varepsilon(t)\in[\mathcal{L}_\varepsilon(t),\mathcal{U}_\varepsilon(t)], \qquad (95)\\
&\qquad D(t)\in[\mathcal{L}_D(t),\mathcal{U}_D(t)]\,\forall\, t\in[0,\tau),\ \varepsilon(t)=D(t)=0\,\forall\, t\in[\tau,T]\Big\},
\end{aligned}$$

where $\mathcal{U}_\varepsilon$ ($\mathcal{U}_D$) denotes an adapted upper bound for the attainable elasticities (durations) and $\mathcal{L}_\varepsilon$, $\mathcal{L}_D$ the corresponding adapted lower bounds. We assume that $\mathcal{L}_\varepsilon < \mathcal{U}_\varepsilon$ and $\mathcal{L}_D < \mathcal{U}_D$. As the dynamics of the firm value and the short rate are not further specified we cannot exclude situations where the elasticity is negative. Analogously, the durations can be positive or negative. Note that in a complete market we would have $\mathcal{U}_\varepsilon = \mathcal{U}_D = +\infty$ and $\mathcal{L}_\varepsilon = \mathcal{L}_D = -\infty$.

Since in the present portfolio problem there are two sources of risk, which are modelled by a two-dimensional Brownian motion, in general an investor needs at least two different risky securities to attain a given combination of elasticity and duration. Nevertheless, two securities might not be sufficient to replicate such a combination. This is due to the restrictions on the attainable elasticities and durations. If an investor can only split up her wealth in a money market account and a corporate bond, she will not in general be able to replicate a given combination of elasticity and duration. In this case the set $\mathcal{A}'_B(0, x_0)$ has to be further restricted via the condition

$$D = \varepsilon \cdot D_C / \varepsilon_C. \tag{96}$$

As a consequence, the investor may only choose the elasticity ε of her portfolio. The duration is then given by condition (96). Alternatively, she can decide for a duration, but then the elasticity is automatically fixed. In the present setting, this missing degree of freedom is characteristic for each portfolio problem with only one risky investment opportunity. Therefore it will be beneficial to the investor if her investment opportunity set is widened by a tradable stock or a subordinated bond.[14] However, if we assume that a firm issued senior and junior bonds, it has to be taken into account, that then we have $V = S + B^{junior} + B^{senior}$.

We are now in the position to solve the portfolio problems.

Proposition 4.2 (Optimal sensitivities in the Briys-de Varenne model) *Consider the portfolio problem (93) and the Briys-de Varenne model with the generalized stochastic short rate (83). Let* $\varepsilon_{uc} := \lambda_V / \sigma_V^2$ *and* $D_{uc} := (\zeta_r - \zeta_V \sigma_r / \sigma_V)/b$ *be the unconstrained optimal elasticity and duration.*

(i) If the investor can invest in at least two corporate securities, as e.g. stock and bond, the optimal elasticity and the optimal duration prior to default can be represented by

$$\varepsilon^*|_{[0,\tau]} = \varepsilon_{uc} + \frac{b \cdot (-\chi_1 + \chi_2) + \sigma_r \cdot (\chi_3 - \chi_4)}{b\sigma_V^2}, \tag{97}$$

$$D^*|_{[0,\tau]} = D_{uc} + \frac{b\sigma_r \cdot (\chi_1 - \chi_2) + (\sigma_V^2 + \sigma_r^2) \cdot (-\chi_3 + \chi_4)}{b^2\sigma_V^2}, \tag{98}$$

where χ_1, χ_2, χ_3, $\chi_4 \geq 0$ *denote Lagrangian multipliers corresponding to the constraints* $\mathcal{U}_\varepsilon - \varepsilon \geq 0$, $\varepsilon - \mathcal{L}_\varepsilon \geq 0$, $\mathcal{U}_D - D \geq 0$, $D - \mathcal{L}_D \geq 0$. *The precise results for all possible combinations of binding constraints are summerized in table 3.*

(ii) If the investor can only invest in one corporate security with elasticity ε_C *and*

duration D_C then the optimal elasticity prior to default, i.e. $t \leq \tau$, is given by

$$\tilde{\varepsilon}(t) = \begin{cases} \mathcal{L}_\varepsilon(t), & \text{if } \mathcal{L}_\varepsilon(t) > \frac{\lambda(t)+\zeta_r(t)b(t)R(t)}{\sigma_V^2(t)+(\sigma_r(t)+b(t)R(t))^2}, \\ \frac{\lambda(t)+\zeta_r(t)b(t)R(t)}{\sigma_V^2(t)+(\sigma_r(t)+b(t)R(t))^2}, & \text{if } \mathcal{U}_\varepsilon(t) \geq \frac{\lambda(t)+\zeta_r(t)b(t)R(t)}{\sigma_V^2(t)+(\sigma_r(t)+b(t)R(t))^2} \geq \mathcal{L}_\varepsilon(t), \\ \mathcal{U}_\varepsilon(t), & \text{if } \mathcal{U}_\varepsilon(t) < \frac{\lambda(t)+\zeta_r(t)b(t)R(t)}{\sigma_V^2(t)+(\sigma_r(t)+b(t)R(t))^2}, \end{cases} \tag{99}$$

where $R := D_C/\varepsilon_C$. The optimal duration is then given by (96).

[Insert table 3]

Proof. (i) For an admissible control (ε, D) the solution of the wealth equation reads as

$$\begin{aligned} X(T) &= x_0 \exp\Big(\int_0^T r(s)\,ds\Big) \cdot \exp\Big(\int_0^T \lambda(s)\varepsilon(s) + \zeta_r(s)b(s)D(s) \qquad (100)\\ &\quad -0.5\sigma_V^2(s)\varepsilon^2(s) - 0.5\big(\sigma_r(s)\varepsilon(s) + b(s)D(s)\big)^2\,ds\Big) \\ &\quad \exp\Big(\int_0^T \sigma_V(s)\varepsilon(s)\,dW_V(s) + \int_0^T \sigma_r(s)\varepsilon(s) + b(s)D(s)\,dW_r(s)\Big) \end{aligned}$$

Hence, we get

$$\begin{aligned} \mathrm{E}(\ln X(T)) &= \ln(x_0) + \mathrm{E}\Big(\int_0^T r(s)\,ds\Big) + \mathrm{E}\Big(\int_0^T \lambda(s)\varepsilon(s) + \zeta_r(s)b(s)D(s) \\ &\quad -0.5\sigma_V^2(s)\varepsilon^2(s) - 0.5\big(\sigma_r(s)\varepsilon(s) + b(s)D(s)\big)^2\,ds\Big), \qquad (101) \end{aligned}$$

Obviously, if the integrand of the second integral is maximized ω-wise with respect to $\varepsilon(t)$ and $D(t)$ we obtain the optimal elasticity and duration. Hence, at time $t < \tau$ we face a constrained optimization problem with a concave objective function[15]

$$\begin{aligned} f(\varepsilon, D) &:= \lambda\varepsilon + \zeta_r bD - 0.5\sigma_V^2\varepsilon^2 - 0.5\big(\sigma_r\varepsilon + bD\big)^2 \qquad (102)\\ &= 0.5(\zeta_V^2 + \zeta_r^2) - 0.5(\zeta_V - \sigma_V\varepsilon)^2 - 0.5(\zeta_r - \sigma_r\varepsilon - bD)^2 \end{aligned}$$

under the constraints $\mathcal{U}_\varepsilon - \varepsilon \geq 0$, $\varepsilon - \mathcal{L}_\varepsilon \geq 0$, $\mathcal{U}_D - D \geq 0$, $D - \mathcal{L}_D \geq 0$. As a consequence, the optimal elasticity and duration have to meet the following Kuhn-Tucker conditions (KTC)

$$\lambda - (\sigma_V^2 + \sigma_r^2)\varepsilon - \sigma_r bD - \chi_1 + \chi_2 = 0 \tag{103}$$

$$b\zeta_r - \sigma_r b\varepsilon - b^2 D - \chi_3 + \chi_4 = 0 \tag{104}$$

$$\chi_1(\mathcal{U}_\varepsilon - \varepsilon) = 0, \tag{105}$$

$$\chi_2(\varepsilon - \mathcal{L}_\varepsilon) = 0, \tag{106}$$

$$\chi_3(\mathcal{U}_D - D) = 0, \tag{107}$$

$$\chi_4(D - \mathcal{L}_D) = 0, \tag{108}$$

with the Lagrangian multipliers $\chi_1,\ \chi_2,\ \chi_3,\ \chi_4 \geq 0$. Assume first that $\chi_i = 0$, $i = 1, 2, 3, 4$, (case 1). Then from the first and second KTC we get $\varepsilon^* = \lambda_V/\sigma_V^2$ and $D^* = (\zeta_r - \sigma_r \lambda_V/\sigma_V^2)/b$. If $\chi_1 > 0$, we conclude $\varepsilon^* = \mathcal{U}_\varepsilon$ and $\chi_2 = 0$. Assume further that $\chi_3 = 0$ and $\chi_4 = 0$ (case 2). Solving the second KTC for D leads to $D^* = (\zeta_r - \sigma_r \mathcal{U}_\varepsilon)/b$. The first KTC can be used to check that χ_1 is really strict positive. Assuming $\chi_1 > 0$ and $\chi_3 > 0$ leads to $\varepsilon^* = \mathcal{U}_\varepsilon$, $D^* = \mathcal{U}_D$ and $\chi_2 = 0$, $\chi_4 = 0$ (case 3). Using the first and the second KTC one can check that $\chi_1,\ \chi_3$ are really strictly positive. The other cases summerized in table 3 can be treated similarly. Besides, the representations (97) of ε^* and D^* can be calculated by solving the first and second KTC for the elasticity and the duration.

(ii) Inserting (96) in the function f of (i) leads to an optimization problem for $\varepsilon(t)$. Solving this problem leads to the optimal elasticity $\tilde{\varepsilon}$. □

Remark:

a) Note that the unconstrained optimal elasticity ε_{uc} does not depend on a traded asset. Hence, the elasticity ε^* only depends on the traded assets, if one of the bounds is touched, whereas the elasticity $\tilde{\varepsilon}$ is never independent of the traded asset.

b) Assuming a logarithmic utility function leads to a separation of the accumulation factor $\int_0^T r(s)\, ds$. This becomes obvious in equation (101). Therefore we can avoid to specify the term structure model further.

Of course, the formulation of proposition 4.2 is not the one which is suitable for practical purposes as Lagrangian multipliers are no observable input parameters. It is always possible to determine from the observable variables in which case of proposition 4.2 we are. However, a detailed such formulation would lead to an enormous number of cases and subcases which does not allow for a compact presentation. To demonstrate at least one such situation in an explicit form we concentrate on case 2 and 3 of the proposition. The other cases mentioned in table 3 can be interpreted in a similar way. Assume that all parameters are positive. If the investor is forced to take less firm value risk than she wants, i.e. $\varepsilon_{uc} > \mathcal{U}_\varepsilon$, and the constraints of the duration are not binding (case 2), her optimal duration D^* equals $(\zeta_r - \sigma_r \mathcal{U}_\varepsilon)/b$, i.e. $D^* > D_{uc}$. Hence, she tries to compensate the restriction of the elasticity by increasing her interest rate exposure. More generally, given an optimal duration ε^* her optimal duration reads as $(\zeta_r - \sigma_r \varepsilon^*)/b$, i.e. a forced decrease of the elasticity leads to an increase of the optimal duration. But this adjustment of the duration is only possible, if the upper bound $\mathcal{U}_D$ is not reached. Otherwise, the corner solution

$\varepsilon^* = \mathcal{U}_\varepsilon$ and $D^* = \mathcal{U}_D$ is optimal (case 3). In figure 1 and 2 we have illustrated the two cases. Note that the maximal interest rate exposure equals ζ_r/b which can be interpreted as a risk premium of the money market.[16] An increase of this premium makes the investment in the money market more attractive. Therefore, the investor is willing to take more interest rate exposure. Moreover, given $\varepsilon_{uc} > \mathcal{U}_\varepsilon$ the excess investment in the money market $(\zeta_r - \sigma_r \mathcal{U}_\varepsilon)/b - D_{uc}$ is greater than with a lower risk premium. Clearly, these arguments can be formalized by applying the KTC.

[Insert figure 1]

[Insert figure 2]

We want to stress that in figure 1 and 2 we have assumed that the unconstrained optimum $(\varepsilon_{uc}, D_{uc})$ stays the same, although ζ_r/b is increased in figure 2. Since the straight line describing the substitution effect is given by

$$D = \frac{\zeta_r}{b} - \frac{\sigma_r}{b}\varepsilon, \tag{109}$$

it becomes obvious that the slope $-\sigma_r/b$ has to be increased, too. This can only occur, if ζ_r and σ_r are increased, because a sole increase of b would contradict the assumption that the unconstrained optimum stays the same. As a consequence, the increase of ζ_r provides for the excess investment in the money market, whereas the increase of σ_r is not relevant for the investor.

As before, we proceed by computing the optimal portfolio processes.

Corollary 4.2 (Optimal portfolios in the Briys-de Varenne model)
(i) Given that the investor can trade in the stock S and the corporate bond B and assume that the elasticities and durations meet the condition

$$\varepsilon_S(t) D_B(t) - \varepsilon_B D_S(t) \neq 0 \tag{110}$$

at time $t \in [0, \tau]$. Then the optimal portfolio process $(\pi_S^*(t), \pi_B^*(t))$ at time t is given by

$$\pi_S^*(t) = \frac{\varepsilon^*(t) D_B(t) - \varepsilon_B(t) D^*(t)}{\varepsilon_S(t) D_B(t) - \varepsilon_B(t) D_S(t)}, \qquad \pi_B^*(t) = \frac{\varepsilon_S(t) D^*(t) - \varepsilon^*(t) D_S(t)}{\varepsilon_S(t) D_B(t) - \varepsilon_B(t) D_S(t)}. \tag{111}$$

Otherwise, there exists a constant $c(t) \in \mathbb{R}$ such that the portfolio elasticity can be expressed as a muliple of the portfolio elasticity, i.e.

$$\pi_S(t) D_S(t) + \pi_B(t) D_B(t) = c(t)\Big(\pi_S(t)\varepsilon_S(t) + \pi_B(t)\varepsilon_B(t)\Big). \tag{112}$$

Besides, only the elasticity $\tilde{\varepsilon}(t)$ of proposition 4.2 with $R(t) = c(t)$ is attainable and each portfolio process $(\pi_S^*(t), \pi_B^*(t))$, which leads to this elasticity, is optimal, i.e. the optimal portfolio process is not uniquely determined.

(ii) Given that the investor can only trade in the corporate bond, then the optimal portfolio process is given by

$$\pi_B^*(t) = \frac{\tilde{\varepsilon}(t)}{\varepsilon_B(t)}. \tag{113}$$

Proof.

(i) To compute the optimal portfolio process we have to solve the system of equations

$$\varepsilon^*(t) = \pi_S(t)\varepsilon_S(t) + \pi_B(t)\varepsilon_B(t), \tag{114}$$

$$D^*(t) = \pi_S(t)D_S(t) + \pi_B(t)D_B(t). \tag{115}$$

If condition (110) is met, the determinant of the matrix

$$A(t) := \begin{pmatrix} \varepsilon_S(t) & \varepsilon_B(t) \\ D_S(t) & D_B(t) \end{pmatrix} \tag{116}$$

is not zero. Hence, the inverse of A exists, which leads to

$$(\pi_S^*(t), \pi_B^*(t))' = A(t)^{-1}(\varepsilon^*(t), D^*(t))'. \tag{117}$$

If condition (110) is not met, the row vectors of the matrix are linearly dependent, which yields (112). As a consequence, the investor cannot choose elasticity and duration independently. Optimizing with respect to the elasticity gives the same result as in proposition 4.2 with $R(t) = c(t)$. Note that $c(t)$ is equal to the ratio of the portfolio's duration and elasticity. Clearly, the optimal portfolio process is not uniquely determined, because the investor can put her money in two corporate securities to match the elasticity $\tilde{\varepsilon}$.

(ii) To match the elasticity $\tilde{\varepsilon}$ the investor has to choose the fraction π_B so that $\tilde{\varepsilon}(t) = \pi_B(t) \cdot \varepsilon_B(t)$. This gives the result. □

5 Conclusion

In this paper we considered portfolio problems with defaultable securities. The default risk was modelled in the firm value framework of Merton (1974), Black/Cox (1976), and Briys/de Varenne (1997). As there the price of a corporate security is given by the price of a contingent claim, we actually face a portfolio problem with derivatives. Apart from this an additional restriction comes into play, which can be interpreted as an accounting equation. This is due to the fact that in firm value approaches the capital structure of the firm is explicitly modelled. As a consequence, the investor can buy at most one stock and one bond, which lead to stochastic bounds on the fractions. Actually, these bounds even depend on the control itself. Using a "small investor assumption" we first omitted the accounting equation and solved the resulting portfolio problem in the Merton model for ease of exposition. Clearly, our approach can be applied to all firm value models where the corporate securities are contingent claims on the firm value modelled by a geometric Brownian motion. Besides, it is valid for a very general class of utility functions. This is due to the fact that we used the elasticity approach to portfolio optimization which proves to be very helpful when derivatives belong to the investment opportunity set.

In a next step we investigated the corresponding constrained portfolio problems. We restricted our considerations to an investor who maximizes her terminal wealth with respect to a logarithmic utility function. The reason is that the bounds on the fractions are difficult to handle. Although we focused on the models of Merton (1974), Black/Cox (1976), and a generalized version of Briys/de Varenne (1997) - in contrast to Briys/de Varenne (1997) we used a general one-factor model for the short rate -, our approach can be applied to other firm value models such as Geske (1977), Longstaff/Schwartz (1995), or Saa-Requejo/Santa-Clara (1999). The study of the constrained problem for more general utility functions is an aspect of future research.

There is another approach to model default risk called the reduced-form approach, which is developed in Jarrow/Turnbull (1995). Formally, default is triggered, when a jump of a Poisson process occurs, which normally is not linked to a theoretical construct like the firm value. Therefore there is not an explicit connection between default and the capital structure of the firm. Hence, portfolio optimization using a reduced-form model can be handled similar to an ordinary portfolio problem in a jump-diffusion framework. To consider this framework in more detail is another aspect of future research.

Footnotes

1) To our knowledge only Merton (1971) considers a portfolio problem with defaultable bonds, but he used a bond model which can be seen as a rudimentary reduced form model with deterministic interest rates.

2) For a derivation of the wealth equation see e.g. Korn/Korn (2001).

3) The word "static" emphasizes that ε only equals the elasticity of the portfolio if π is held constant. Otherwise, Ito's rule has to be applied and additional terms come into play.

4) In the next section, we will discuss this point in detail.

5) Another closed-form solution is derived for options guaranteed by fixed margins. See Johnson/Stulz (1987) for further details.

6) See e.g. Fleming/Soner (1993) or Korn/Korn (2001).

7) See Korn/Kraft (2001).

8) In Black/Cox (1976) there is a typo in their formula (8) for the bond value. In their notation, the seventh term of their formula (8) should be $y^{\theta+\eta}\mathcal{N}(z_7)$ instead of $y^{\theta-\eta}\mathcal{N}(z_7)$. As a consequence, the last four terms in (8) cancel out if the dividend yield is equal to zero. Note that we tried to adopt the notation of Black and Cox, but we changed the sign of z_3.

9) See e.g. Jones/Mason/Rosenfeld (1984), Kim/Ramaswamy/Sundaresan (1993), or Longstaff/Schwartz (1995).

10) See e.g. Korn/Korn (2001).

11) Briys/de Varenne (1997) also allow for deviations of the absolute priority rule. We do not consider this generalization here, because it does not fundamentally alter our results.

12) In the context of stochastic interest rates duration was introduced by Cox/Ingersoll/Ross (1979) using their interest rate model. They define duration as $G^{-1}(-D_C)$, where G is a model dependent function. This is due to the fact that duration is normally measured in units of time. Only, if the interest rates are deterministic or the time-continuous Ho-Lee model is used both definitions coincide. Since in our context only the sensitivity C_r/C becomes relevant, we use the word "duration" for this sensitivity in accordance with the deterministic case.

13) See Merton (1973) or Duffie (2001) for a textbook reference.

14) We want to emphasize that a complete market does not always complete the

investor's opportunity set. This can be due to institutional restrictions, as e.g. faced by bond investment fonds, which are not permitted to invest in stocks.

15) For notational convenience we omitted the dependencies on time t.

16) Money market is used in a rather sloppy way. Here we mean a market which is purely affected by the interest rate risk.

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Table 1

Probability of not-touching the barrier for a portfolio problem with stocks and bonds. This table reports the probability that an investor who can put her wealth into bonds and stocks of a firm does not want to buy the whole issues. The investor maximizes her terminal wealth with respect to a logarithmic utility function. It is assumed that the drift of the firm value is 0.15, the volatility of the firm value is 0.2, and the riskless interest rate is 0.05. The first row reports the different investment horizons T. The following rows report the probabilities for different ratios of initial wealth x_0 to inital firm value v_0. It becomes obvious that for small investors the probability is almost one.

Time	Ratio x_0/v_0										
T	0.4	0.35	0.3	0.25	0.2	0.15	0.1	0.05	0.01	0.005	0.001
1	0	0.301	0.612	0.853	0.971	0.998	≈ 1	≈ 1	≈ 1	≈ 1	≈ 1
3	0	0.153	0.338	0.545	0.748	0.906	0.985	≈ 1	≈ 1	≈ 1	≈ 1
5	0	0.107	0.241	0.402	0.586	0.774	0.926	0.995	≈ 1	≈ 1	≈ 1
7	0	0.083	0.188	0.319	0.479	0.662	0.847	0.977	≈ 1	≈ 1	≈ 1
10	0	0.062	0.141	0.243	0.373	0.537	0.732	0.926	0.999	≈ 1	≈ 1

Table 2

Probability of not-touching the barrier for a portfolio problem with corporate bonds. This table reports the probability that an investor who can only put her wealth into corporate bonds does not want to buy the whole issue. The investor maximizes her terminal wealth with respect to a logarithmic utility function. It is assumed that the drift of the firm value is 0.15, the volatility of the firm value is 0.2, the riskless interest rate is 0.05, the inital firm value v_0 is 1000, and the face value of the corporate bond is 750. The first row reports the different investment horizons T. The following rows report the probabilities for different ratios of initial wealth x_0 to inital firm value v_0. It becomes obvious that even for relative small investors the probability is far from being one.

Time	Ratio x_0/v_0				
T	0.01	0.005	0.0001	0.00005	0.00001
1	0.151	0.271	0.523	0.617	0.782
3	0.076	0.123	0.226	0.273	0.372
5	0.042	0.066	0.124	0.143	0.207
7	0.022	0.040	0.070	0.090	0.126
10	0.008	0.017	0.032	0.042	0.059

Table 3

Optimal elasticity and duration in the Briys-de Varenne model. This table reports the optimal elasticity and duration of a portfolio problem where an investor can put her wealth in at least two different corporate securities. These securities are modelled by a Briys-de Varenne model. Since one has to consider upper and lower bounds for the attainable elasticities and durations nine cases have to be distinguished. These cases are characterized via the Lagrangian multipliers. For notational convenience we omit the dependencies on time t.

case	Lagrange multipliers				optimal elasticity	optimal duration
no.	χ_1	χ_2	χ_3	χ_4	ε^*	D^*
1	$=0$	$=0$	$=0$	$=0$	λ_V/σ_V^2	$(\zeta_r - \sigma_r\lambda_V/\sigma_V^2)/b$
2	>0	$=0$	$=0$	$=0$	$\mathcal{U}_\varepsilon$	$(\zeta_r - \sigma_r\mathcal{U}_\varepsilon)/b$
3	>0	$=0$	>0	$=0$	$\mathcal{U}_\varepsilon$	$\mathcal{U}_D$
4	>0	$=0$	$=0$	>0	$\mathcal{U}_\varepsilon$	$\mathcal{L}_D$
5	$=0$	>0	$=0$	$=0$	$\mathcal{L}_\varepsilon$	$(\zeta_r - \sigma_r\mathcal{L}_\varepsilon)/b$
6	$=0$	>0	>0	$=0$	$\mathcal{L}_\varepsilon$	$\mathcal{U}_D$
7	$=0$	>0	$=0$	>0	$\mathcal{L}_\varepsilon$	$\mathcal{L}_D$
8	$=0$	$=0$	>0	$=0$	$(\lambda - \sigma_r b\mathcal{U}_D)/(\sigma_V^2 + \sigma_r^2)$	$\mathcal{U}_D$
9	$=0$	$=0$	$=0$	>0	$(\lambda - \sigma_r b\mathcal{L}_D)/(\sigma_V^2 + \sigma_r^2)$	$\mathcal{L}_D$

Figure 1

Unconstrained optimal elasticity not attainable. This figure illustrates the situation of an investor who is not able to attain the unconstrained optimal elasticity ε_{uc}, but the bounds on the duration are not binding. The quadrat restricts the set of attainable combinations of elasticity and duration. Her optimal strategy is given by the intersection of the quadrat and the straight line $D = \zeta_r/b - \varepsilon\sigma_r/b$, because duration is not restricted. As a consequence, she can take all the additional interest risk that she wants to take.

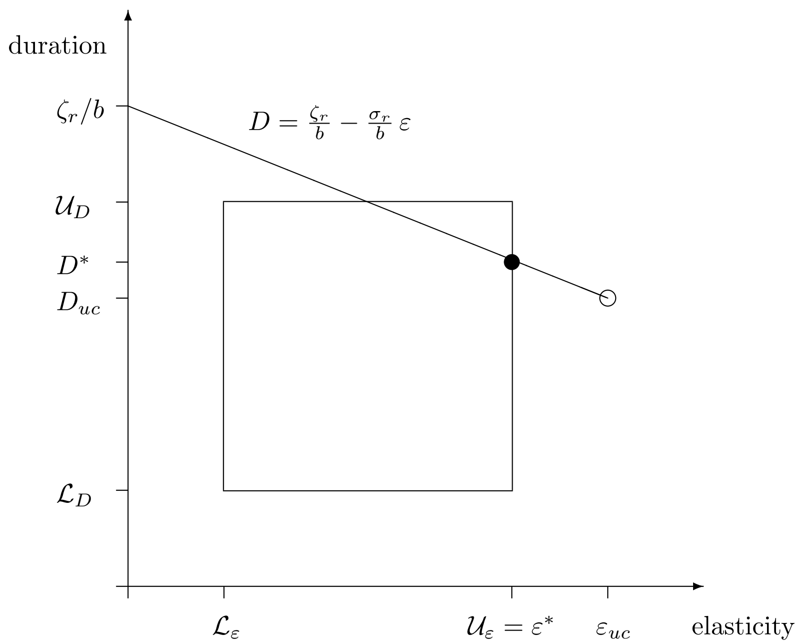

Figure 2

Unconstrained optimal elasticity and duration not attainable. This figure illustrates the situation of an investor who is neither able to attain the unconstrained optimal elasticity ε_{uc} nor the unconstrained optimal duration D_{uc}. This is due to the fact that both the upper bound of the elasticity and the upper bound of the duration are violated. The quadrat restricts the set of attainable combinations of elasticity and duration. Since the straight line $D = \zeta_r/b - \varepsilon\sigma_r/b$ does not cut the quadrat she cannot take all the additional interest rate risk that she wants to take. Hence, the right upper corner of the quadrat represents the optimal combination of elasticity and duration.