Hedging in Complete and Incomplete Markets

by

Siegfried Trautmann

Johannes Gutenberg-Universität Mainz

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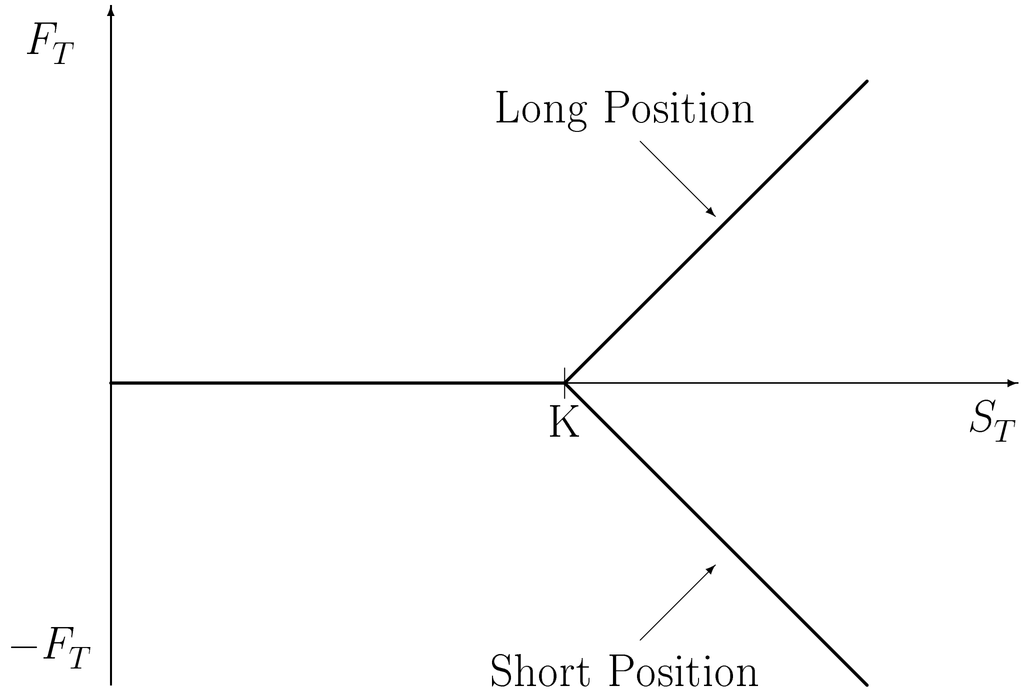
2. Hedging without Shortfall-Risk in Complete Markets: Delta-Hedging
3. Hedging without Shortfall-Risk in Incomplete Markets: Super-Hedging
4. Variance-Minimizing Hedging Strategies in Incomplete Markets
5. Shortfall Risk-Minimizing Hedging Strategies
6. Conclusion

1. Introduction

Hedging Object: Short position of a European call with exercise price K, expiration date T, terminal value $F_T = (S_T - K)^+$, and present value F_0.



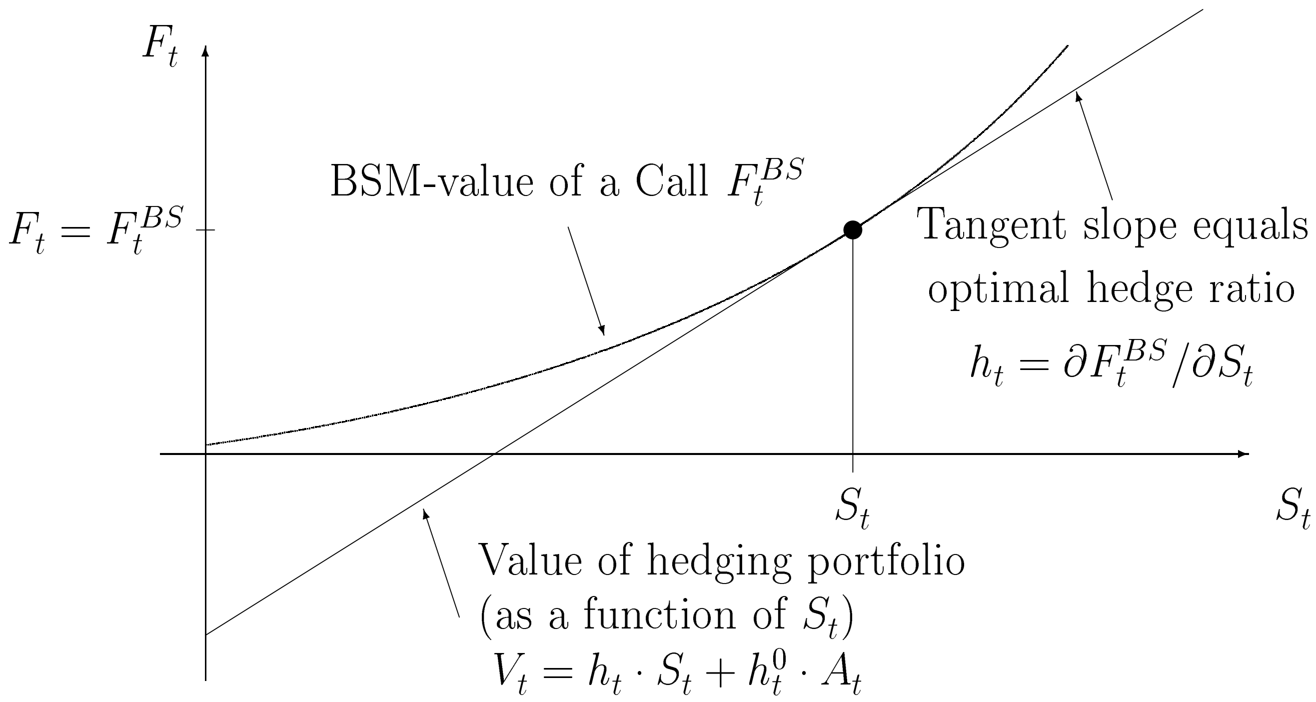
Hedging Instruments: Underlying stock with value S_t and money market account with value $A_t = \exp(rt)$. To be hedged partially or perfectly, the hedger chooses a predictable strategy $H = (h, h^0)$ where h denotes the number of stocks and h^0 denotes the the number of money market accounts. If H is self-financing we have $V_t = V_0 + \int h_u dS_u + \int h_u^0 dA_u$ or in discrete-time models, $V_t = V_0 + \sum_{u=1}^{t}(h_u \cdot \Delta S_u + h_u^0 \cdot \Delta A_u)$.

Hedging Error: In incomplete markets there does in general not exist a self-financing strategy H with $V_T = F_T$. When implementing a hedging strategy the increment of the hedging error (additional cash inflows or cash withdrawals) is defined as follows: $dC_t \equiv dF_t - dV_t = dF_T - h_t dS_t - h_t^0 dA_t$. (If incremental values are due to „price jumps“ then dX is replaced by ΔX).

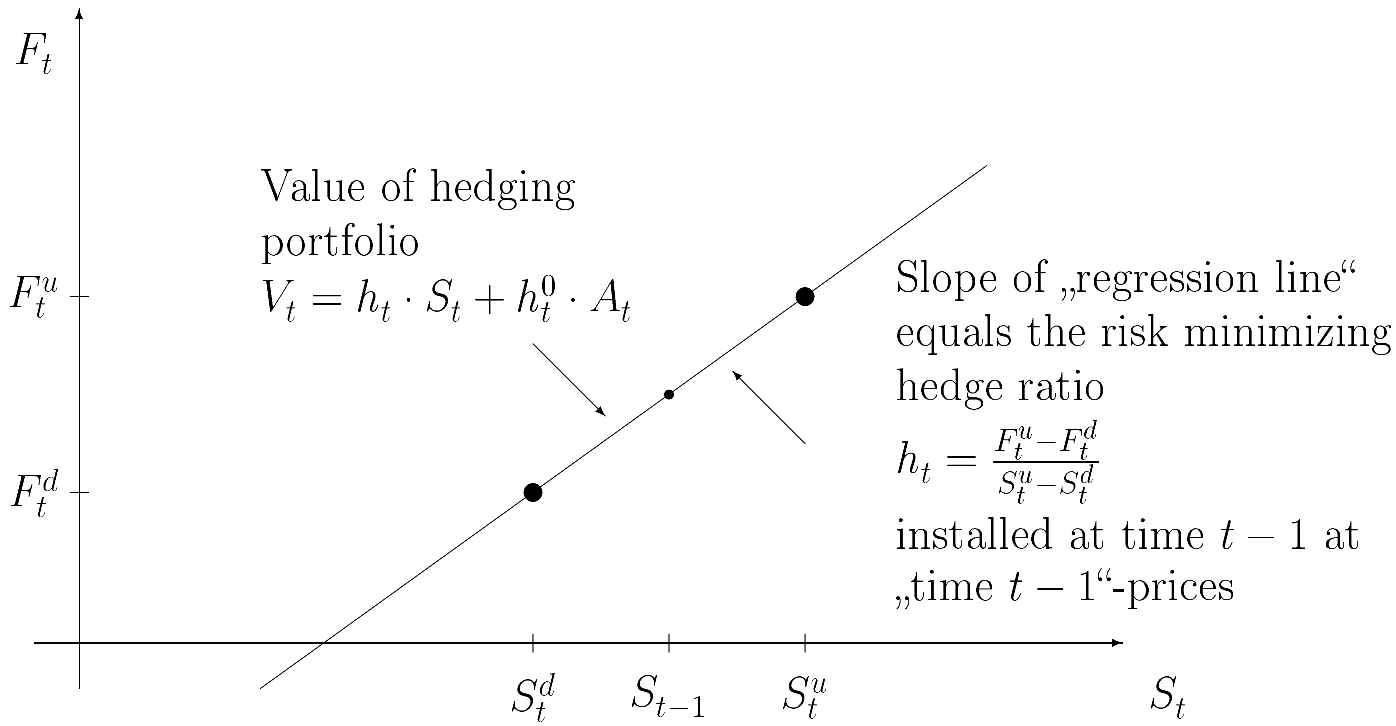
Even in complete markets there does not exist a self-financing strategy $H = (h, h^0)$ with $V_T = F_T$ if the hedger is only willing to invest $\bar{V}_0 < F_0$ at time zero.

Hedging Shortfall: $(F_T - V_T)^+$, that is, the additional cash inflow necessary at terminal date $t = T$ to replicate the written claim.

2. Hedging without Shortfall-Risk in Complete Markets: Continuous-time model of Black/Scholes/Merton (1973)



Perfect Hedges in Complete Markets: Discrete-time Model of Cox/Ross/Rubinstein (1979)



Fundamental Theorems of Asset Pricing

(under mild conditions, even for infinitely many securities)

Theorem 1: Asset prices F_t^i do not admit profitable arbitrage if and only if there is a probability measure Q on $(\Omega, \mathcal{F})$ equivalent to the physical measure P for which for each asset i the process of relative prices

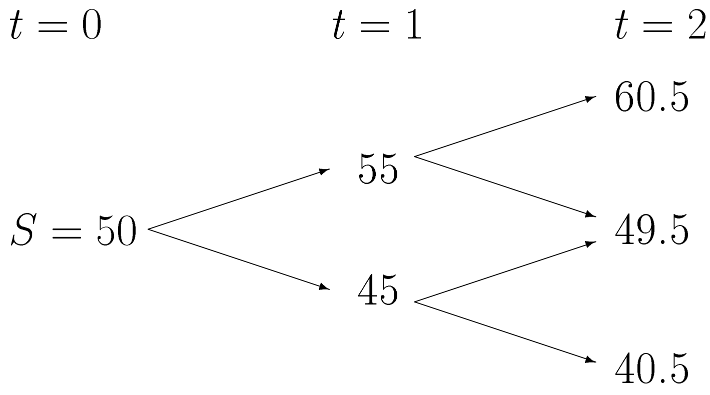
$$Z_t^i \equiv F_t^i / A_t \quad \text{is a Q-martingale}$$

where A_t specifies the price of the money market account at t.

Theorem 2: If the market is complete (that is there is a trading strategy with traded assets which replicates arbitrary claims) then there exists at most one equivalent martingale measure Q.

Example 1(Perfect hedge in a binomial model)

Process parameters U = 1.1, D=0.9, interest rate r = 0%, martingale probability q=(1-D)/(U-D) = 0.5.



Replicating strategy for a European Call with strike price K=45, maturity date T=2 and fair value $F_t = E_Q(F_T \mid \mathcal{F})$:

$$t = 1: \qquad h_1(55) = \tfrac{15{,}5-4{,}5}{60{,}5-49{,}5} = 1 \qquad F_1(55) = \frac{1}{2}\cdot 15,5 + \frac{1}{2}\cdot 4,5 = 10$$

$$h_1(45) = \tfrac{4{,}5-0}{49{,}5-40{,}5} = 0,5 \qquad F_1(45) = \frac{1}{2}\cdot 4,5 = 2,25$$

$$t = 0: \qquad h_0 = \tfrac{10-2{,}25}{55-45} = 0,775 \qquad F_0 = \frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 2,25 = 6,125$$

Approaches to hedge a short position in a Contingent Claim

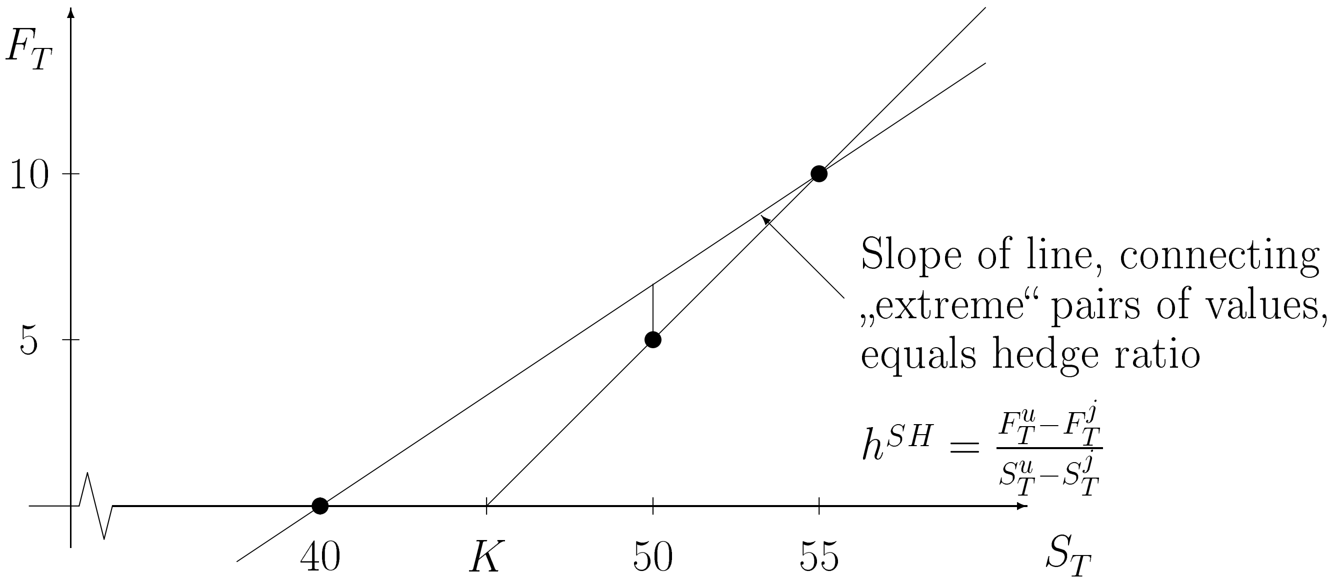
<table>
<tr><th></th><th>Complete Markets</th><th>Incomplete Markets</th><th></th></tr>
<tr><td>No Shortfall Risk</td><td>Delta-Hedging:
Black/Merton/Scholes (1973)
Cox/Ross/Rubinstein (1979)</td><td>Superhedging:
El Karoui/Quenez (1995)
Naik/Uppal (1992)</td><td rowspan="2">No Restriction on Initial Hedging Capital</td></tr>
<tr><td rowspan="3">Shortfall Risk</td><td></td><td>Local Risk-Hedging:
Föllmer/Schweizer (1991)
Schweizer (1992)
Grünewald/Trautmann (1997)</td></tr>
<tr><td colspan="2">Global Variance-Hedging: Schweizer (1996)</td><td>————————</td></tr>
<tr><td colspan="2">Shortfall Probability-Hedging:
Föllmer/Leukert (1999)
Expected Shortfall-Hedging:
Föllmer/Leukert (1998), Cvitanić/Karatzas (1998)
Cvitanić (1998), Schulmerich/Trautmann (1999)</td><td>Restriction on Initial Hedging Capital</td></tr>
</table>

3. No Shortfall-Risk in Incomplete Markets: Superhedging

A strategy $H \in \mathcal{H}$ is called a superhedging strategy if $V_T(H) \geq F_T$ P-a.s.
The time t cost of carrying out the cheapest superhedging strategy is given by the supremum of the expected terminal value over all Equivalent Martingale Measures $Q \in \mathcal{Q}$:

$$\inf_{H \in \mathcal{H}} \{V_t(H) \mid V_T(H) \geq F_T \text{ P-a.s.}\} = \sup_{Q \in \mathcal{Q}} E_Q(F_T \mid \mathcal{F}_t)/A_{T-t}$$

Example 2 (Superhedging in a trinomial one-period model)



4. Risk- and Variance-Minimizing Hedging Strategies in Incomplete Markets

The strategy minimizing **Local Risk** (LR) was introduced by Schweizer (1991) and Föllmer/Schweizer (1991). The LR-definition is very technical. Intuitively we can characterize it in the following way:

Minimize for each t in time the expected quadratic hedging error for the next instant:

$$E((dC)^2) = E((dF - hdS - h^0 dA)^2).$$

This is tantamount to minimizing for each instant the variance of the hedging error conditional on the mean hedging error being zero:

$$\min d\langle C\rangle \quad \text{conditional on } E(dC) = 0\,.$$

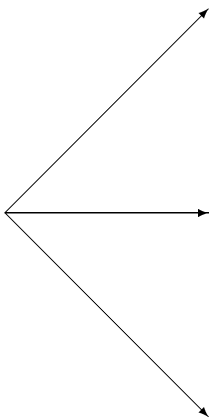
4.2 Local Risk-Hedging in the Trinomial Model

Example 3 (One-period trinomial Model)

Process parameters U = 1,1, D=1, J=0,8, interest rate r=0%.

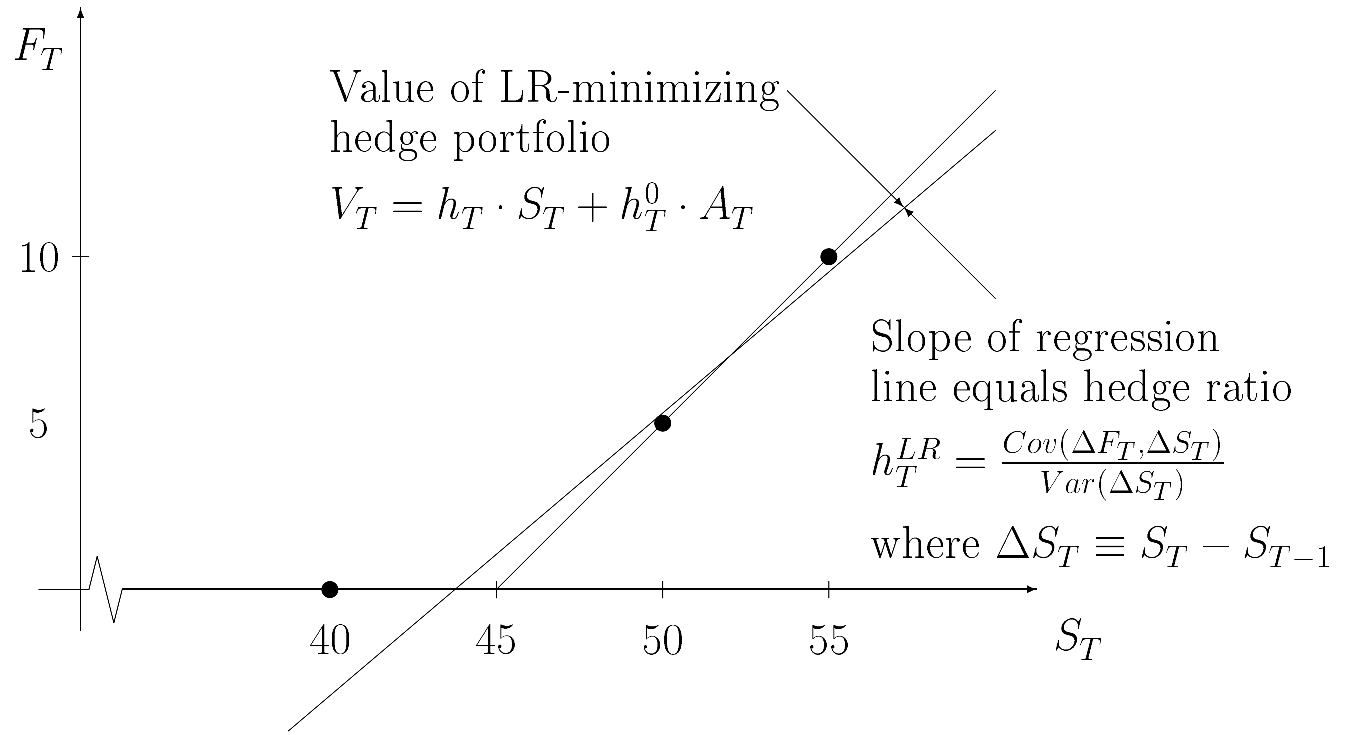
With two traded assets (stock and money market account) we get the following Equivalent Martingale Measures (EMMs):

$S = 50$



Stock	Call	Physical Measure	EMMs
S_T	F_T	P	Q
55	10	0,57	$2q(\omega_j)$
50	5	0,42	$1-3q(\omega_j)$
40	0	0,01	$0 < q(\omega_j) < 1/3$

Local Risk-Hedging in the Trinomial Model: A graphical illustration



4.3 Local Risk-Hedging in the Jump-Diffusion Model

Stock Price Dynamics:

$$dS_t = \alpha S_{t^-}dt + \sigma_D S_{t^-}dW_t + S_{t^-}(I_t dN_t - \lambda k dt)$$

$$S_t = S_0 \exp\left\{(\alpha - \frac{1}{2}\sigma_D^2 - \lambda k)t + \sigma_D W_t + \sum_{i=1}^{N_t} \ln(1 + L_i)\right\} .$$

where

α	$\equiv$	the constant instantaneous drift of the total process,
W	$\equiv$	a standard Brownian Motion, σ_D is the constant volatility of the diffusion,
N_t	$\equiv$	Poisson Process with parameter λ,
T_i	$\equiv$	arrival time of jump i, $i = 1, 2, \ldots N_t$,
L_i	$\equiv$	percentage size of jump i, $i = 1, 2, \ldots N_t$ where $I_{T_i} = L_i$. The jump in the return, $ln(1 + L_i)$, is normally distributed with mean $\alpha_J - \frac{1}{2}\sigma_J^2$ and variance σ_J^2. (It follows $L_i > -1$.)
k	$\equiv$	$E(L_i)$ expected percentage jump size.

Theorem (GRÜNEWALD/TRAUTMANN (1996))*In the Poisson jump diffusion model with lognormally distributed jump sizes in the stock price the locally risk-minimizing hedge ratio equals*

$$\begin{aligned} h^{LR} &= (1-\gamma)F_s^{LR} + \gamma \frac{E_L(\Delta S \Delta F^{LR})}{E_L((\Delta S)^2)} \\ &= (1-\gamma)\sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{n,l} N(d_1(n,l)) + \gamma \frac{\lambda}{\sigma_{jump}^2} \sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{n,l} \left\{ -k \frac{F^{BS}(S,K,r_{n,l},\sigma_n,\tau)}{S} + \ldots \right. \end{aligned}$$

This hedge ratio has a nice interpretation as a local Beta-coefficient:

$$h^{LR} = \frac{d\langle F^{LR}, S\rangle}{d\langle S, S\rangle} = \frac{\mathrm{Cov}(dF^{LR}, dS|\mathcal{F}_{-})}{\mathrm{Var}(dS|\mathcal{F}_{-})}$$

.

Numerical Analysis of LR-Hedging in a Jump-Diffusion Model

Grünewald/Trautmann (1996) find

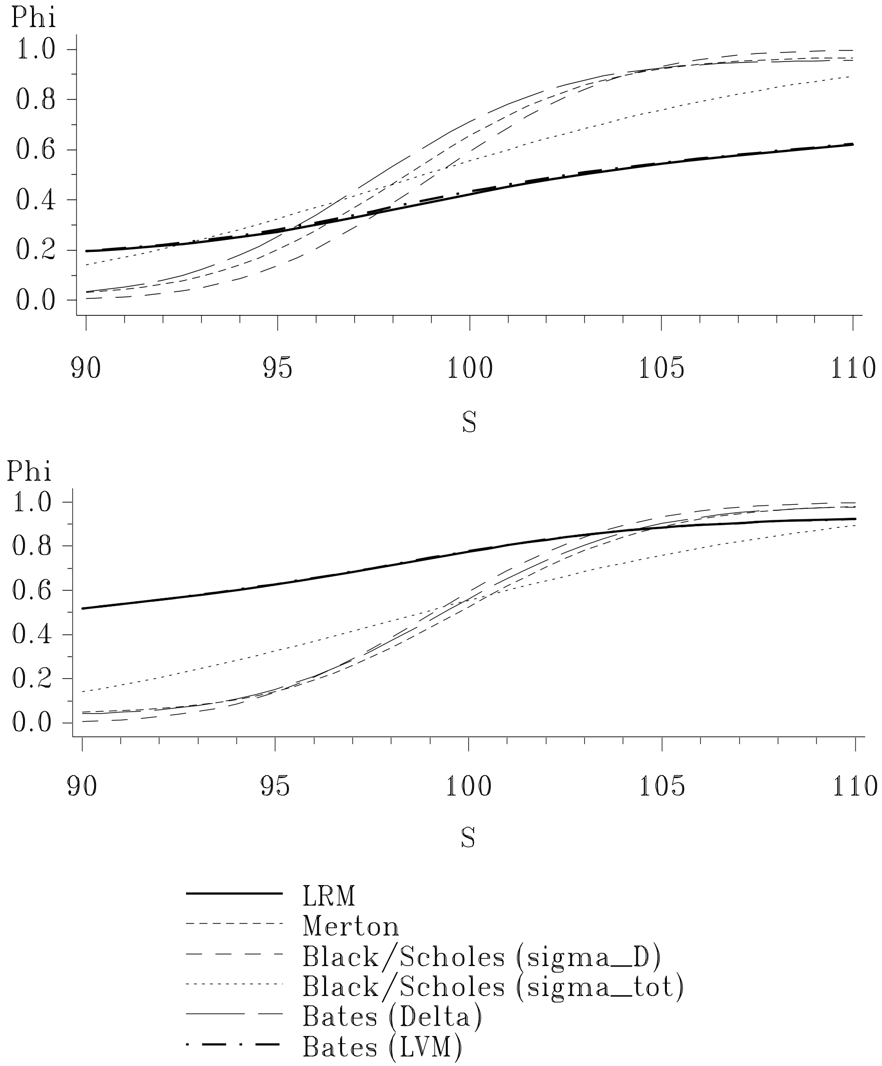
- that the LR-minimizing and LV-minimizing strategies show the lowest sensitivity of the *hedge ratios* with respect to a change in the stock price. Consequently discrete rebalancing of the portfolio and transaction costs matter less with these strategies.
- that if the expected jump size, k, is upward (downward), the LR-minimizing and LV-minimizing hedge ratios are higher (lower) thus anticipating the expected jump size.
- that the LR-minimizing and LV-minimizing strategies show superior worst case behaviour: extremely high hedging costs are less likely! According to a Monte Carlo Simulation with 10,000 sample paths to determine the *frequency distribution of the total hedging costs* based on the parameters: $S = 100$, $K = 100$, $\lambda = 1$, $k = -0.1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $T = 1$, $\alpha = 0.15$, $r = 0.05$, $R = 1$. The portfolio is rebalanced once a week.

Hedge Ratios as a Function of the Stock Price

Parameters: $K = 100$, $r = 0.1$, $T = 1/12$, $\alpha = 0.15$, $\lambda = 1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $R = 1$.

$k = -0.1$:

$k = 0.1$:



Simulated distribution of total discounted hedging costs

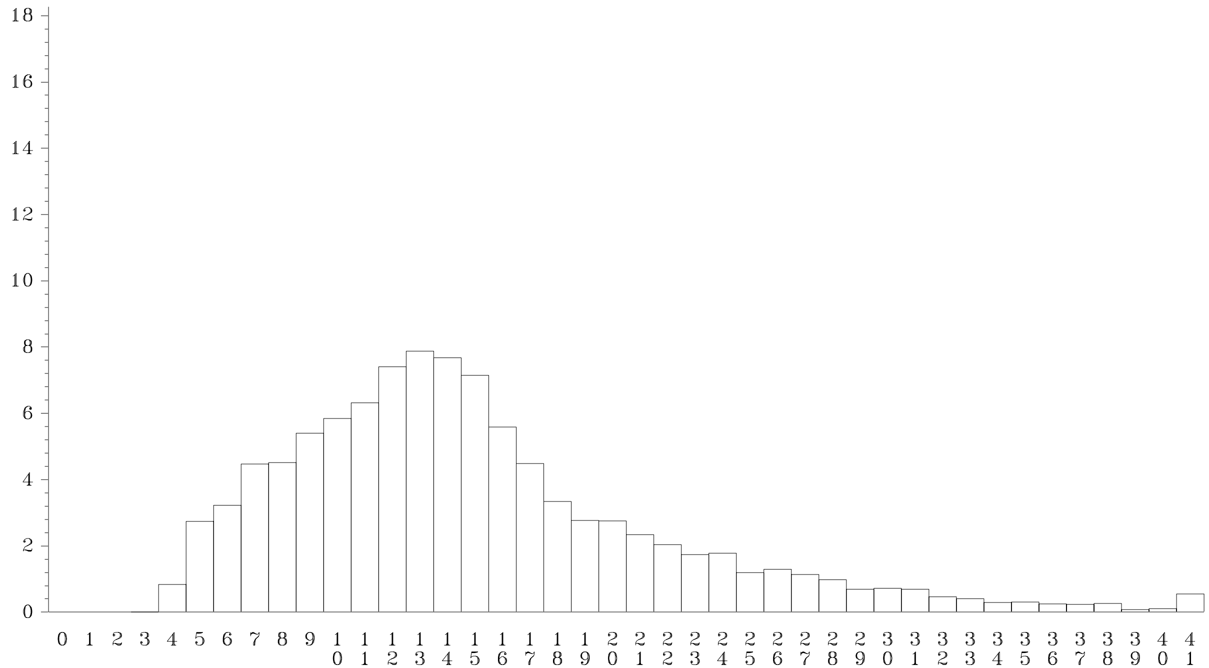
(Parameters: $S = 100$, $K = 100$, $\lambda = 1$, $k = -0{,}1$, $\sigma_{tot} = 0{,}3$, $\gamma = 0{,}8$, $T = 1$, $\alpha = 0{,}1$, $r = 0{,}05$, $R = 1$)

Total costs	LR	GV	Bates (LVM)	Bates (Delta)	Merton	B/S (σ^2_{tot})	B/S (σ^2_D)	Super-hedge	No hedge
(Initial costs)	(14,90)	(14,90)	(16,55)	(16,55)	(14,23)	(14,23)	(8,01)	(100,00)	(0,0)
Mean	14,93	14,93	14,88	14,22	14,24	14,48	14,23	12,81	17,94
Std. deviation	7,05	7,04	7,07	8,88	8,88	8,06	9,50	14,60	23,21
Skewness	1,45	1,66	1,39	1,62	1,71	1,75	1,86	2,02	2,40
Kurtosis	4,24	6,83	3,70	2,47	2,99	3,58	3,80	3,52	10,05
99% Quantile	**37,48**	**37,75**	**37,22**	42,63	43,47	41,87	45,93	64,87	104,96
95% Quantile	**28,57**	**28,29**	**28,66**	33,10	33,43	31,51	34,76	46,78	58,96
90% Quantile	**24,34**	**24,12**	**24,48**	27,95	27,81	26,34	28,65	35,42	45,21
75% Quantile	17,91	17,70	18,06	18,17	18,08	17,83	17,88	15,39	28,30
50% Quantile	13,67	13,79	13,53	10,22	9,95	11,13	8,64	4,88	10,93
25% Quantile	10,20	10,34	10,06	8,12	8,23	9,32	8,04	4,88	0,00
10% Quantile	7,21	7,29	7,19	6,99	7,38	7,75	7,91	4,88	0,00
5% Quantile	5,98	5,95	5,95	6,57	7,07	6,85	7,81	4,88	0,00
1% Quantile	4,59	4,45	4,57	6,16	6,70	5,85	7,59	4,88	0,00

Frequency Distribution of Total Hedging Costs

(Parameters: $S = 100$, $K = 100$, $\lambda = 1$, $k = -0.1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $T = 1$, $\alpha = 0.1$, $r = 0.05$, $R = 1$)

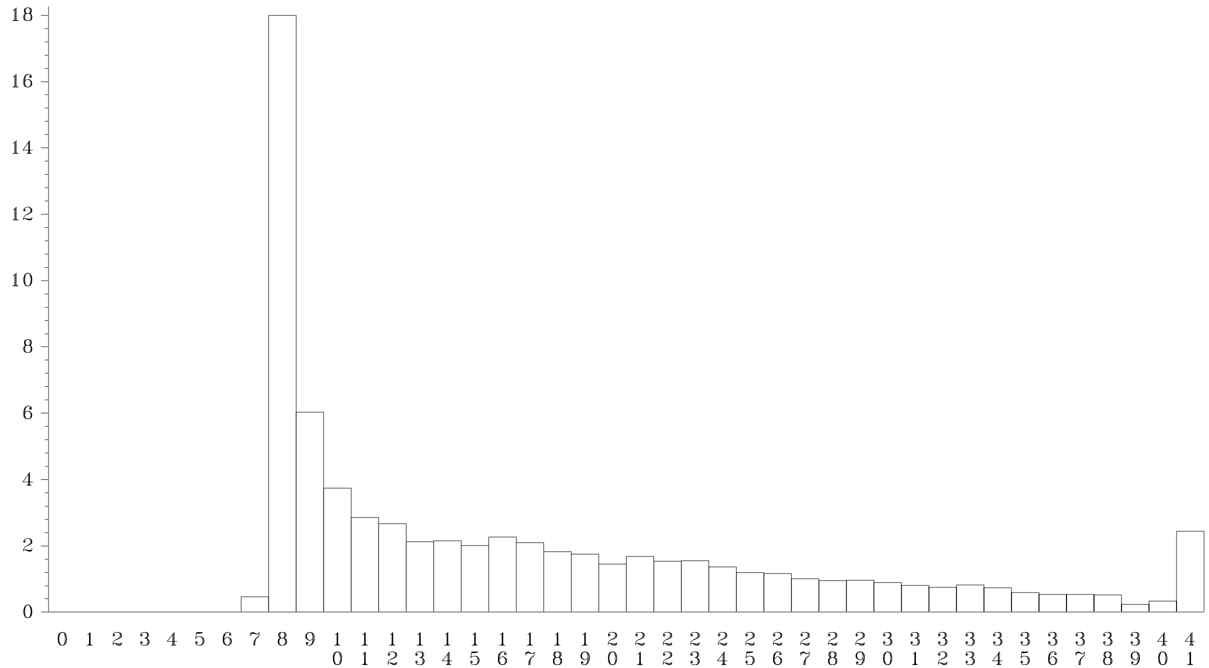
Panel A: Locally Risk-minimizing strategy



Frequency Distribution of Total Hedging Costs

(Parameters: $S = 100$, $K = 100$, $\lambda = 1$, $k = -0.1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $T = 1$, $\alpha = 0.1$, $r = 0.05$, $R = 1$)

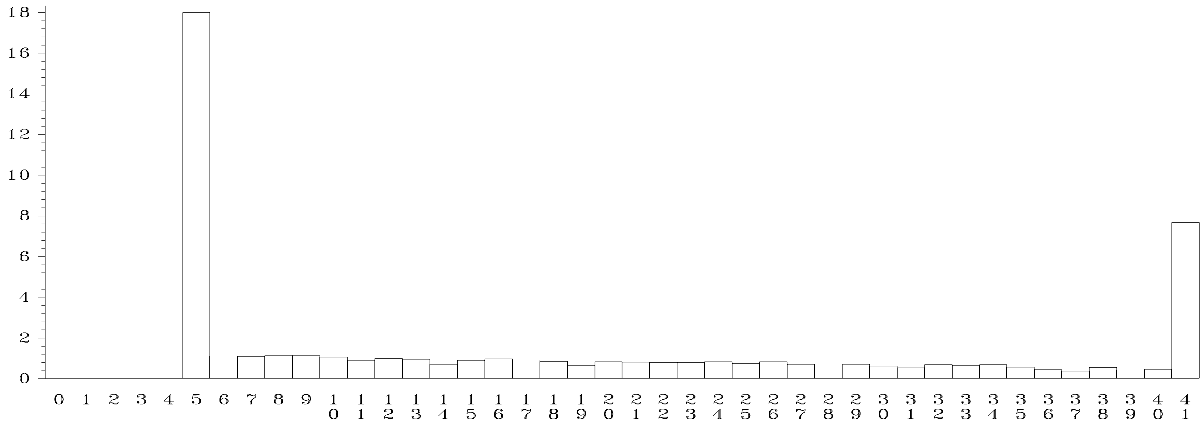
Panel G: Black/Scholes strategy (σ_D)



Frequency Distribution of Total Hedging Costs

(Parameters: $S = 100$, $K = 100$, $\lambda = 1$, $k = -0.1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $T = 1$, $\alpha = 0.1$, $r = 0.05$, $R = 1$)

Panel H: Superhedging strategy



5. Shortfall Risk-Minimizing Hedge Strategies

Motivation:

- In complete markets: hedger is not willing to invest completely the proceeds from writing the option.
- In incomplete markets: hedger is not willing or able to finance a superhedging strategy.

Measures of Shortfall Risk:

- Shortfall Probability (not a coherent risk measure, $\rightarrow$ Quantile Hedging)
- Expected Shortfall (coherent risk measure)

5.1 Two-Step Procedure

In accordance with the *martingale approach to portfolio optimization* the following two-step procedure is suitable:

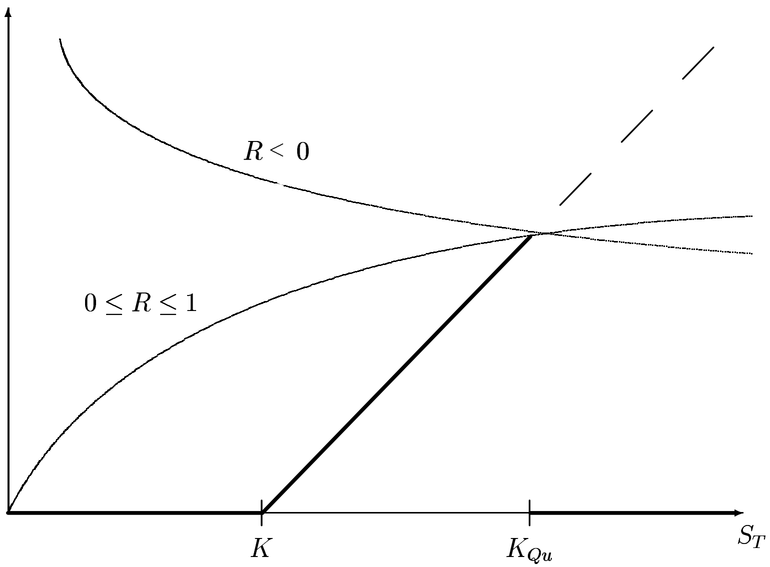
(1) Calculation of a modified contingent claim which is attainable given that the initial hedging capital $\bar{V}_0$ is less than F_0.

(2) Superhedging of the modified contingent claim

5.2 Quantile-Hedging: Minimizing the Shortfall Probability

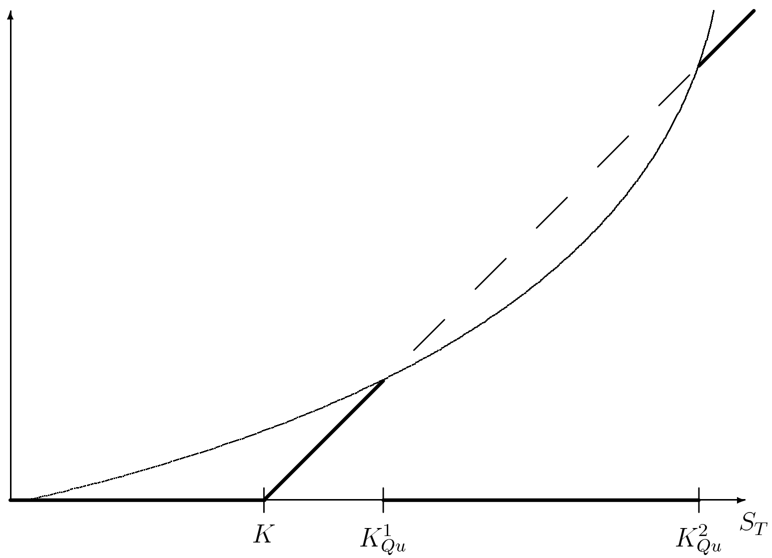
- Modification factor $\varphi(\cdot)$ of the written call to be hedged *corresponds* to the optimal test (function) according to the Neyman-Pearson-Lemma.
- Modified contingent claim of a call in a jump-diffusion model:

Relative Risk Aversion R$\leq$ 1



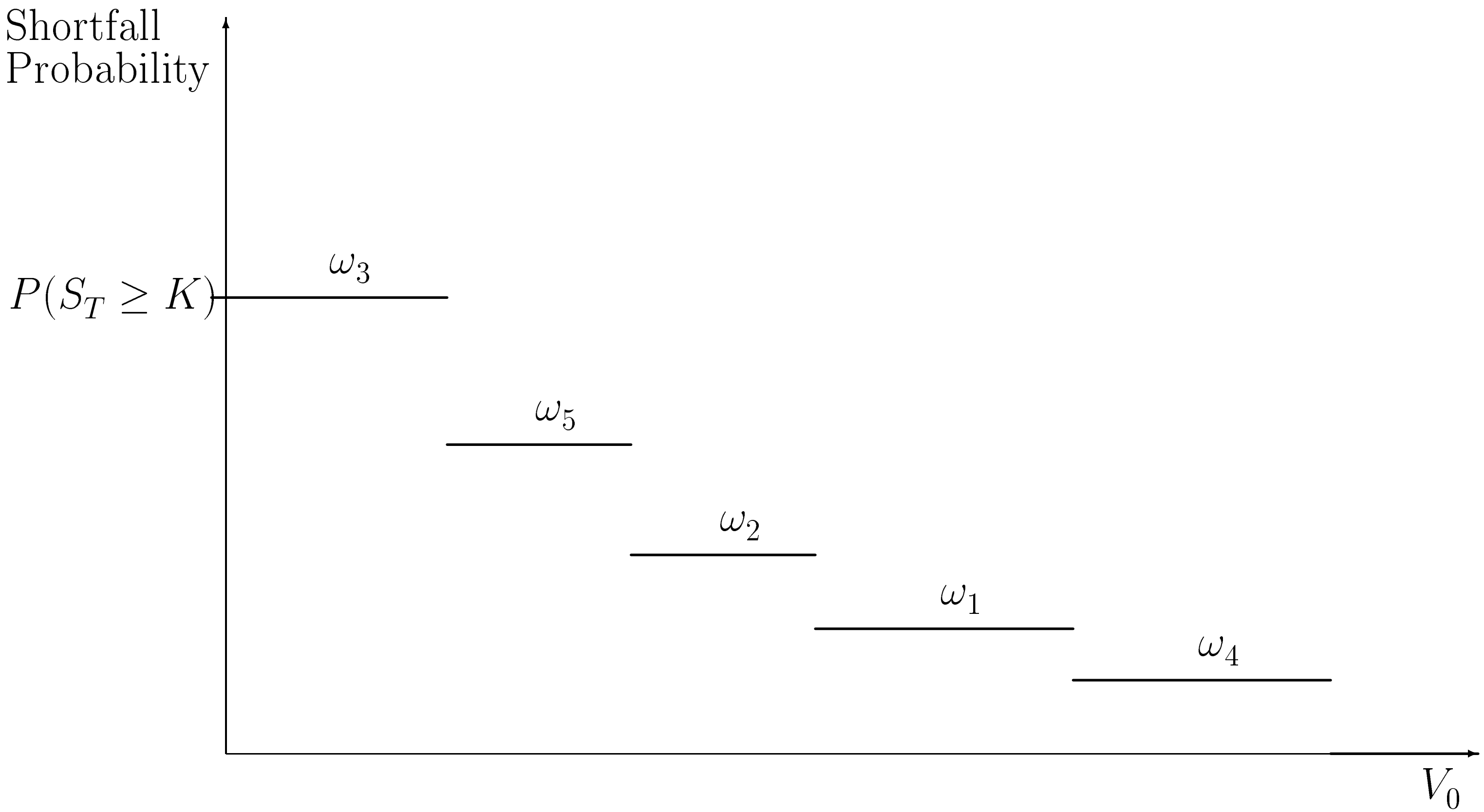
Truncated Option

Relative Risk Aversion R>1



Truncated + Gap Option

Warning: In the discrete analogon (multinomial setting) the optimal modified contingent claim cannot be found via the Neyman-Pearson-Lemma.



5.3 Expected Shortfall-Hedging

Theorem (Föllmer/Leukert (1998)) *If $\mathcal{Q}$ is a singleton then the optimal modified contingent claim has the following representation:*

$$X_T^* = \varphi^* F_T = \left(\mathbf{1}_{\left\{ \frac{dP}{dQ} > c_{ES} \right\}} + \gamma \mathbf{1}_{\left\{ \frac{dP}{dQ} = c_{ES} \right\}} \right) F_T$$

where

$$c_{ES} = \inf \left\{ c | E_Q \left(\mathbf{1}_{\left\{ \frac{dP}{dQ} > c \right\}} F_T \right) \leq \bar{V}_0 A_T \right\}$$

and

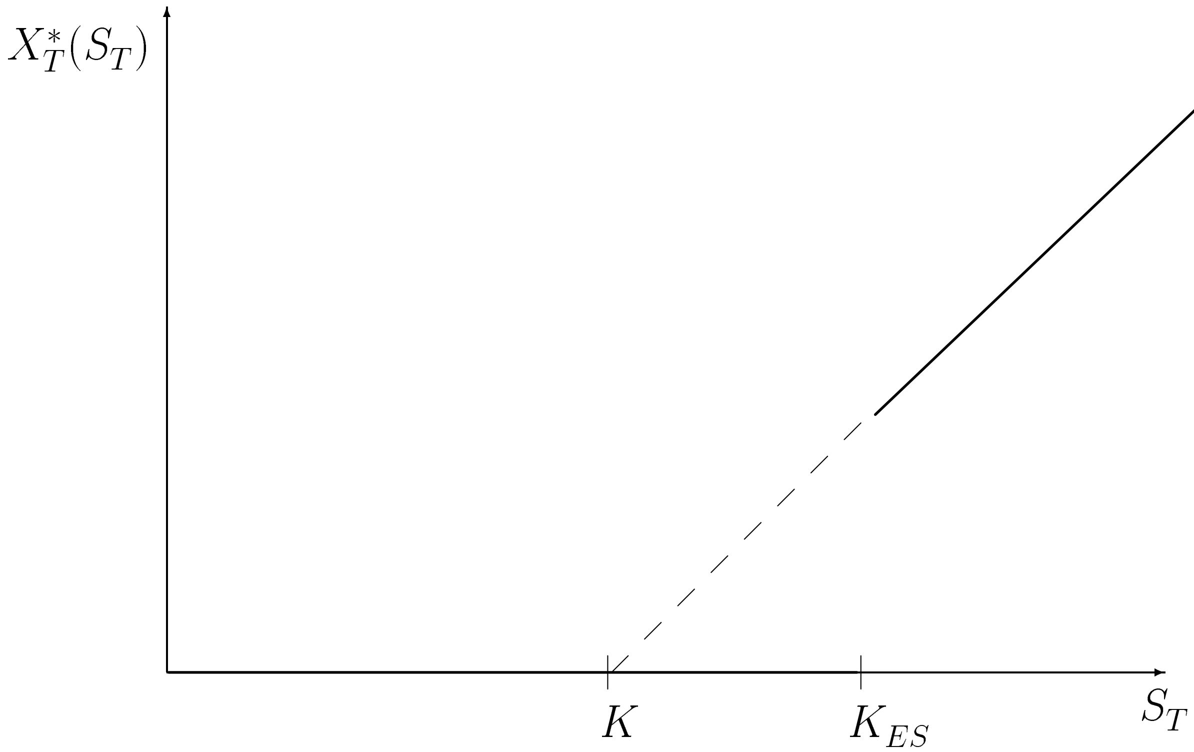
$$\gamma = \left(A_T \bar{V}_0 - E_Q \left(\mathbf{1}_{\left\{ \frac{dP}{dQ} > c_{ES} \right\}} F_T \right) \right) / \left(E_Q \left(\mathbf{1}_{\left\{ \frac{dP}{dQ} = c_{ES} \right\}} F_T \right) \right).$$

Replicating the modified contingent claim with strategy (V_0^*, H^*) solves the problem of minimizing the expected shortfall under the constraints
$V_0 \leq \bar{V}_0$ and $P(V_T(H) \geq 0) = 1$.

Remark: If $\mathcal{Q}$ is not a singleton, the approach of Föllmer/Leukert provides only a sufficient condition for the optimality of the modified contingent claim.

Expected Shortfall Hedging in a Black/Scholes/Merton-world:

Modified Claim (Gap option with strike K_{ES}) to be hedged perfectly.



5.4 Expected Shortfall-Hedging: Relaxing the constraint $P(V_T(H) \geq 0) = 1$

Theorem (Schulmerich/Trautmann (1999)) *If $\mathcal{Q}$ is a singleton and Ω is finite then the optimal modified contingent claim X_T^* has the following representation:*

$$X_T^* = F_T \mathbf{1}_{\{\frac{dP}{dQ} > c_{ES}\}} + \gamma \mathbf{1}_{\{\frac{dP}{dQ} = c_{ES}\}}$$

where

$$c_{ES} = \min_{\omega \in \Omega} \left\{ \frac{dP}{dQ}(\omega) \right\}$$

and

$$\gamma = \left(A_T \bar{V}_0 - E_Q \left(\mathbf{1}_{\{\frac{dP}{dQ} > c_{ES}\}} F_T \right) \right) / \left(E_Q(\mathbf{1}_{\{\frac{dP}{dQ} = c_{ES}\}}) \right).$$

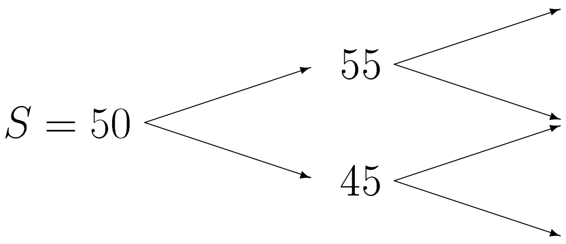
Replicating the modified contingent claim X_T^* with strategy (V_0^*, H^*) solves the problem of minimizing the expected shortfall under the constraint $V_0 \leq \bar{V}_0$.

Remarks:

- Expected Shortfall-Hedging without the constraint $P(V_T(H) \geq 0) = 1$ on the terminal hedging portfolio value might result in a lower expected shortfall (Schulmerich/Trautmann, 1999).
- Even LR-Hedging, ignoring such a constraint, often leads to a lower expected shortfall compared to Föllmer/Leukert's approach.

Example 4 (Expected Shortfall Hedging in the Binomial Model)

Process parameters U = 1,1, D=0,9, $p^u = 0{,}58$, $p^d = 0{,}42$, interest rate r = 0%, strike price K=45 and initial hedging capital $\bar{V}_0 = 4$.



S_T	F_T	P	Q	$V_T(H^*)$ according to F/L	$V_T(H^*)$ according to S/T
60.5	15.5	0.3364	0.25	15.5	15.5
49.5	4,5	0.4872	0.5	0.25	4.5
40,5	0	0.1764	0.25	0	-8.5

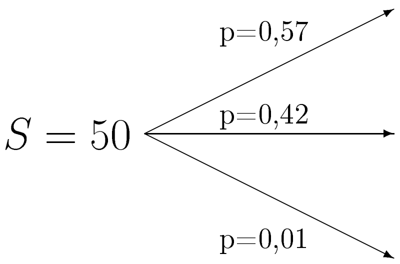
Expected Shortfall:

ESF(F/L): $0.3364 \cdot 0 + 0.4872 \cdot 4.25 + 0.1764 \cdot 0 = 2.07$ (Föllmer/Leukert, 1998)

ESF(S/T): $0.3364 \cdot 0 + 0.4872 \cdot 0 + 0.1764 \cdot 8.5 = 1.5$ (Schulmerich/Trautmann, 1999)

Example 5 (Expected shortfall-hedging in the trinomial model)

Process parameters U = 1.1, D=1, J=0.8, strike price K=45, interest rate r = 0%, initial hedging capital $\bar{V}_0 = 5.275$.

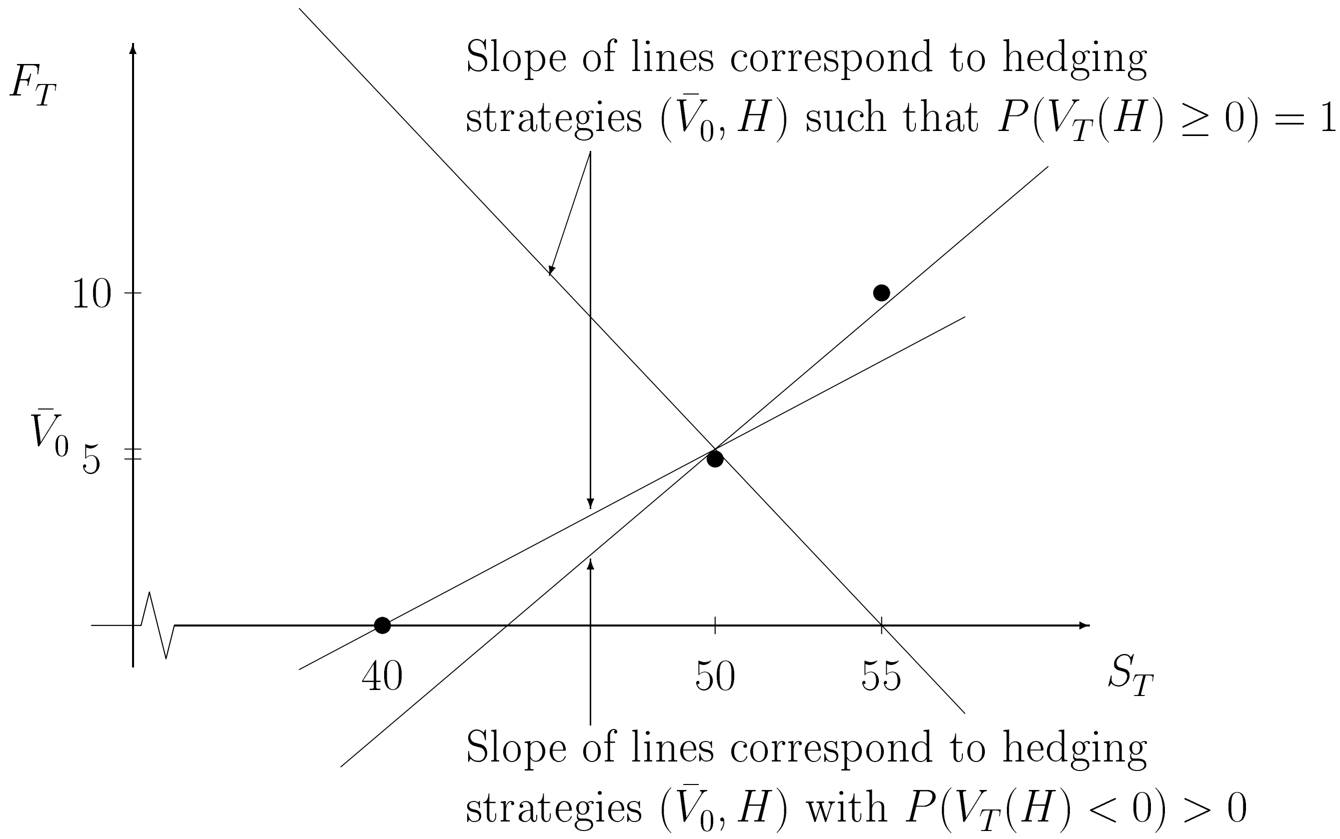


		Terminal value V_T of optimal hedging strategy according to		
S_T	F_T	F/L	S/T	LR
55	10	7,91	10	9.86
50	5	5	5	5.275
40	0	0	-4.175	-3.9

Expected Shortfall:

ESF(F/L):	$0.57 \cdot (10 - 7.91) + 0.42 \cdot 0 + 0.01 \cdot 0 = 1,19$	(Föllmer/Leukert, 1998)
ESF(S/T):	$0.57 \cdot 0 + 0.42 \cdot 0 + 0.01 \cdot (0 + 4.175) = 0.04$	(Schulmerich/Trautmann, 1999)
LR:	$0.57 \cdot (10 - 9.86) + 0.42 \cdot 0 + 0.01 \cdot 3.9 = 0.12$	(Schweizer, 1992)

Illustration:



Recent approaches to Shortfall Hedging

(i.e., Shortfall Probability-Hedging, Expected Shortfall-Hedging, Shortfall-Variance-Hedging)

	Expected shortfall-hedging considered?	Incomplete markets setting included?	Modified claim allowed to take on negative values?	Existence of optimality of (signed) NP-structure proven?
Föllmer/Leukert (1998)	Yes	Yes	**No**	Yes
Cvitanić/Karatzas (1998)	Yes	**No**	Yes	Yes
Cvitanić (1998)	Yes	Yes	**No**	Yes
Cvitanić/Karatzas (1999)	Yes	Yes	**No**	Yes
Pham (1999)	**No**	Yes	Yes	Yes
Schulmerich/Trautmann (1999)	Yes	Yes	Yes	**No**
?/?/?	Yes	Yes	Yes	Yes